Spring 2020

Inductive Learning and Writing Proofs: Student Experiences in Advanced University Mathematics

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Abstract:

Many students encounter formal mathematical proofs for the very first time in college. The process of learning to write these proofs is challenging by nature and often poorly facilitated, imposing a barrier to studying more advanced mathematics.This study investigates the exact nature of this learning process, and in particular the role of inductive learning, to better understand how to improve student experiences. Through interviews with past students of introductory proof-based math courses at Yale, either Math 225 or Math 230, this capstone finds that learning to write proofs is a primarily inductive process.More so than lecture, homework is crucial to this kind of learning, specifically problems where the student needs to consult teaching figures, their peers, or the internet for help. Within the framework of inductive learning, the hints and solutions students gain access to serve as examples of proofs, from which they can identify trends and extract insights. This capstone also investigates the consequences of the inductive teaching method on student behavior, especially when not executed well. Students turn to collaboration and self-learning, but also become heavily dependent on online resources, which harms student confidence. I recommend that instructors establish clearer expectations regarding student learning and provide more opportunities for students to gain exposure to proof examples central to the inductive learning process. This grants students greater agency and empowers them to take control of their learning.

Inductive Learning and Writing Proofs:

Student Experiences in Advanced University Mathematics

Joy Qiu

Education Studies Capstone

Advisors: Richard Hersh and Patrick Devlin

May 1, 2020

Abstract

Many students encounter formal mathematical proofs for the very first time in college. The process of learning to write these proofs is challenging by nature and often poorly facilitated, imposing a barrier to studying more advanced mathematics. This study investigates the exact nature of this learning process, and in particular the role of inductive learning, to better understand how to improve student experiences. Through interviews with past students of introductory proof-based math courses at Yale, either Math 225 or Math 230, this capstone finds that learning to write proofs is a primarily inductive process. More so than lecture, homework is crucial to this kind of learning, specifically problems where the student needs to consult teaching figures, their peers, or the internet for help. Within the framework of inductive learning, the hints and solutions students gain access to serve as examples of proofs, from which they can identify trends and extract insights. This capstone also investigates the consequences of the inductive teaching method on student behavior, especially when not executed well. Students turn to collaboration and self-learning, but also become heavily dependent on online resources, which harms student confidence. I recommend that instructors establish clearer expectations regarding student learning and provide more opportunities for students to gain exposure to proof examples central to the inductive learning process. This grants students greater agency and empowers them to take control of their learning.

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"If problem-solving is the heart of mathematics, then proof is its soul."[1]

Introduction

Age-old wisdom tells us that we learn to write by reading. Research tells us the same thing: although good readers don't always make good writers, those who read tend to write better.[2] After all, it is through reading that we gain exposure to new vocabulary and different conventions, techniques, and styles, and we improve as writers when we learn and are able to integrate these into our writing. This way of acquiring skills and knowledge—by studying examples—is called *inductive* learning.[3] For instance, native language acquisition is largely inductive,[4] and so too is the development of writing style. But this capstone project isn't about writing essays; it is about mathematical proofs. *How do we learn to write those?*

For most of K-12, mathematics isn't (and shouldn't be) taught inductively. Imagine being expected to infer the quadratic formula $\frac{-b\pm\sqrt{b^2-4ac}}{2a}$ by looking at examples of polynomials and their solutions. There, a *deductive* approach, of first learning a rule and then applying it, is more appropriate. However, when a student begins to study advanced mathematics (usually beginning with Linear Algebra or Real Analysis in their first of second year of college), the material takes on a different flavor: mathematical proofs take center stage. A proof, in essence, is a series of logical arguments which show some mathematical claim to be true. This definition and transition are explained further in the literature review, but the key point is that students must suddenly learn proof-writing, a new skill, to engage successfully with advanced university mathematics.

In my personal experience as an undergraduate math major and as a Peer Tutor for an introductory proof-based math course at Yale, this "new skill" has not been taught well. Within the literature, there is resounding consensus that undergraduates have enormous difficulty

[1] Alan Schoenfeld, Mathematician and Professor at the University of California, Berkeley.

[2] Arthur N. Applebee, "Teaching High-Achieving Students: A Survey of the Winners of the 1977 NCTE Achievement Awards in Writing," Research in the Teaching of English 12, no. 4 (1978): 339–48, www.jstor.org/stable/40170740.

[3] Evan Heit, "Properties of Inductive Reasoning," *Psychonomic Bulletin & Review* 7, no. 4 (2000): 569–92, https://link.springer.com/content/pdf/10.3758%2FBF03212996.pdf.

[4] Stephen D. Krashen, *Principles and Practice in Second Language Acquisition*, Language Teaching Methodology Series (New York: Prentice-Hall, 1982), https://pdfs.semanticscholar.org/b9a7/847c076812231f2990b1fb713a1df3e8d2d6.pdf.

understanding and constructing proofs—that this "new skill" is not *learned* well, either.[5] Of note, Yale's pedagogical approach towards teaching proofs is not unique among its peer institutions (it is shared, for example, by Stanford, Princeton, and MIT[6]). For these reasons, I want to take a moment to illustrate how I was taught to write proofs:

As a first-year student, I enrolled in Math 230, an intensive vector calculus and real analysis course for prospective math majors which—importantly—did not assume any knowledge of proofs. During the first lecture, the professor explained three methods of writing proofs and provided an example of each. For direct proof, he proved why the sum of two odd integers is even; for proof by contradiction, why $\sqrt{2}$ is irrational; and for proof by induction, why the sum of the first n integers is equal to $\frac{n(n+1)}{2}$. This single lecture was the only time he explicitly taught about proof-writing the entire semester. I remember staring down at the problems on the first homework: "Let $a < b$ be real numbers. Prove there are infinitely many irrational numbers between a and b." *What?* His three examples had done nothing to prepare me to approach this problem, let alone to write a coherent proof, and in my desperation I turned to Google and haphazardly reproduced the most upvoted answer on math.stackexchange.com. For the rest of the semester, this was the process I endured: struggle with a problem for a few hours, concede I was unable to solve it, wrangle the solution from the TA or peer tutor or a friend or the internet, attempt to understand the provided solution, inevitably fail to understand small portions, write down everything in the best way I could; rinse and repeat.

Yet three years later, despite having only received that single lecture of proof-writing instruction, I feel like I know how to write a proof. I've mastered the syntax; I understand the flow the logic; I've gained intuition and feel comfortable meandering along different paths until uncertainty resolves to a solution. I have spent two years tutoring other students in an introductory proof-based math course and will be graduating with a mathematics degree in May, and so some might call my professor's "sink or swim" approach to teaching effective. Certainly,

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[5] Keith Weber, "Student Difficulty in Constructing Proofs: The Need for Strategic Knowledge," *Educational Studies in Mathematics* 48 (2001): 101–19, https://link.springer.com/content/pdf/10.1023%2FA%3A1015535614355.pdf.
[6] Stanford's Math 113 is a Linear Algebra which includes proof-writing in its course objectives; MIT's only proofs course is a 2-week long transition course offered during IAP; Princeton's only transition course, MAT214, is "not recommended for prospective math majors or applied majors."

I learned proofs somewhere along the way. But if he didn't explicitly teach me, how did I learn? What did I learn? And did I learn it inductively?

Now, as the peer tutor for Math 225, an introductory proof-based Linear Algebra course at Yale, I see my students struggle with the same things I did. Many have expressed their frustration when trying to approach a problem or logically structure a proof. Bolder students come to me with a solution they've found online, and ask me to walk through the steps they don't understand. When I hear students tell me they "feel like [they] still don't know how to write a proof" or won't be taking further math coursework because their experience with proofs has stolen their confidence, I'm reminded of the imposter syndrome and mental complexes I had as a first year. I wanted so desperately to learn, yet week after week I would find myself grappling with a guilty conscience and what felt like my own overwhelming incompetence. That semester, I contemplated dropping both the class and math major about twice a week, probably. All of this makes me wonder: How did I manage to surmount those insecurities through my learning? Is this experience shared? And of course, how can we make it better?

There exists a rich body of literature documenting the difficulties that students have with mathematical proofs. Learning to write proofs is hard, and various studies have shown that students struggle to identify, reproduce, understand, write, and produce proofs,[7] with these challenges arising from factors like inadequate cognitive development, a misunderstanding of socio-mathematical norms, notional difficulties, ineffective proof strategies, and more.[8] For instance, by surveying math students at the University of Cordoba, Recino & Gordono found that a majority of students were unable to determine whether a given paragraph of mathematical statements constituted a proof.[9] By evaluating proofs about group homomorphisms written by students in an undergraduate abstract algebra course, Weber discovered his students were unable to produce a coherent mathematical argument almost 70% of the time.[10] By studying reading

[7] Guerson Harel and Larry Sowder, "Toward Comprehensive Perspectives on the Learning and Teaching of Proof," *Second Handbook of Research on Mathematics Teaching and Learning*, January 2007, https://math.ucsd.edu/~harel/TowardComprehensivePerspective.pdf.

[8] Keith Weber, "Research Sampler 8: Students' Difficulties with Proof," Mathematical Association of America, accessed December 10, 2019, https://www.maa.org/programs/faculty-and-departments/curriculum-department-guidelines-recommendations/teaching-and-learning/research-sampler-8-students-difficulties-with-proof.

[9] Angel M Recio and Juan D Godino, "Institutional and Personal Meanings of Mathematical Proof," *Educational Studies in Mathematics* 4, no. 1 (2001): 83–99.

[10] Weber, "Student Difficulty in Constructing Proofs: The Need for Strategic Knowledge."

behavior and tracking eye movement, Alcock et al. demonstrated that students have a weaker ability to identify logical relationships between statements in a proof compared to expert mathematicians, leading to lower proof comprehension.[11] These findings, along with the thousands of proof studies in the literature, are usually derived from experiments which focus on one specific class or a few specific proofs, and then measure learning outcomes within that single context. There has been no study, however, which identifies the inductive nature of the learning process and asks how it impacts the student experience. There has also been no study which makes central the student experience, which seeks to illustrate and reveal the narrative underlying how students understand and perceive their own learning.

With regards to pedagogy, while many studies have proposed interventions for teaching proofs (such as by incorporating visual software,[12] delivering the lecture in a specific style,[13] or using diagrams to represent logic[14]), the majority are intended to increase proof comprehension, not to build or improve proof-writing skills. They offer solutions that respond directly to the challenges they document students facing, but do not postulate how they expect students to learn in the first place. Furthermore, no study explicitly names "inductive learning" as a core component of the learning process, or wishes to capitalize on this aspect to teach proofs better.

This capstone project begins to fill these voids in the literature and investigate many of the intuitions about the learning process that I've formed from my personal experiences. By understanding how exactly students use deductive and inductive learning when learning to write proofs, we can gain insight on the shortfalls of the pedagogy currently used in many advanced university mathematics courses. In addition, while my experiences pointed me strongly towards inductive learning, I did not want to presuppose that the learning process is largely inductive.

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[11] Lara Alcock et al., "Investigating and Improving Undergraduate Proof Comprehension," Notices of the American Mathematical Society 62, no. 1, (May 11, 2015), https://repository.lboro.ac.uk/articles/Investigating_and_improving_undergraduate_proof_comprehension/9369368.

[12] Gila Hanna, "Proof, Explanation and Exploration: An Overview," *Educational Studies in Mathematics* 44 (2000): 5–23, https://link.springer.com/content/pdf/10.1023%2FA%3A1012737223465.pdf.

[13] Mika Gabel and Tommy Dreyfus, "Affecting the Flow of a Proof by Creating Presence—a Case Study in Number Theory," *Educational Studies in Mathematics* 96, no. 2 (October 1, 2017): 187–205, https://doi.org/10.1007/s10649-016-9746-z; Mika Gabel and Tommy Dreyfus, "The Flow of a Proof - The Example of the Euclidean Algorithm," in *Proceedings of the 37th International Conference for the Psychology of Mathematics Education*, vol. 2 (Kiel, Germany: PME, 2013), 321–28, http://lettredelapreuve.org/pdf/PME37/Gabel.pdf.

[14] Gabriel J. Stylianides and Andreas J. Stylianides, "Research-Based Interventions in the Area of Proof: The Past, the Present, and the Future," *Educational Studies in Mathematics* 96, no. 2 (October 2017): 119–27, https://doi.org/10.1007/s10649-017-9782-3.

Through interviews with Yale undergraduate math students, I sought to establish that this was indeed true. This capstone investigates the power of deductive and inductive learning for writing mathematical proofs, and crafts a narrative of student learning which prioritizes students' own reflections on proofs. Especially in a field that sees so much dropout, understanding this experience and the implications it has for pedagogy is critical for making mathematics more accessible.

Research Questions

In the context of students enrolled in advanced university math courses:

1. How do students learn to write mathematical proofs, and what is the role of inductive learning in this process?
    a. How do students develop mastery over different components of proofs?
    b. How do different course resources support and enable inductive learning?
2. How does the inductive teaching method shape course culture and impact students, beyond just their learning?

Scope

This is a worthwhile topic to study if and only if proofs are important. My capstone project is not concerned with whether anyone who is learning mathematics needs to or should learn proofs. The internet is filled with persuasive essays written by mathematics professors to convince their students of the necessity of proofs,[15] many of which appeal to the importance of reasoning and problem solving. These are valid reasons, but they do not serve as motivation for my project. I agree with the literature that proofs and proof-writing are a core component of advanced mathematics. Indeed, because all advanced university math courses are built around proofs, it is a given that proofs are necessary to learn in order to understand mathematics in the way that mathematicians would engage with the subject. Therefore, for a student to realistically pursue math-related majors in college, mastering proof-writing and developing confidence in that

[15] Joshua N Cooper, "Why Do We Have to Learn Proofs!?" (University of South Carolina Department of Mathematics, n.d.), http://people.math.sc.edu/cooper/proofs.pdf; James Hurley, "What Are Mathematical Proofs and Why Are They Important" (University of Connecticut Department of Mathematics, n.d.), https://www2.math.uconn.edu/~hurley/math315/proofgoldberger.pdf.

ability in an early introductory course is critical. The ability to construct proofs well is important because failure to do so becomes an unnecessary barrier to learning and continued study.[16] To teach proofs better is to make higher-level mathematics and related STEM fields more accessible.

On that note, this capstone project is not foremost focused on the issue of accessibility in advanced mathematics. While it is a significant personal motivator for my interest in the project, and I discussed issues like imposter syndrome at length in the introduction, I am not trying to holistically evaluate the ways that proofs impact accessibility. Such a study would necessarily involve questions surrounding gender and racial identity and socioeconomic status, which mine does not. I believe enough research has been done on the impact of these factors to STEM learning outcomes at the elementary and secondary levels, that a study conducted at the tertiary level would be unlikely to reveal anything surprising; I imagine I would find that the effects are similar but exacerbated at higher levels of education. The perspective I take is through the lens of inductive learning, and I try to understand how this pedagogical technique can positively impact students both in their learning outcomes and overall experience.

[16] Annie Selden and John Selden, "Overcoming Students' Difficulties in Learning to Understand and Construct Proofs," in *Making the Connection*, ed. Marilyn P. Carlson and Chris Rasmussen (Washington DC: The Mathematical Association of America, 2008), 95–110, https://doi.org/10.5948/UPO9780883859759.009.

Methodology

The findings in this capstone are predicated on the stories and experiences shared by 12 Yale College students. Importantly, they are not based on my observations of student behavior, but rather students' recollections and perceptions of themselves and their learning. In certain ways, this approach means my analysis exists two layers of interpretation and storytelling above reality. But rather than being a limitation, this methodology is very much intentional: This capstone's primary purpose is to illuminate and center student narratives in the discussion of learning proof-writing, and obtaining a perfect reflection of reality is less pertinent to this aim than understanding how students believe they experienced this reality.

I conducted semi-structured interviews with 12 current Yale College students who had taken or were currently enrolled in an introductory proof-based math course at Yale, either Math 225 or Math 230 (formally, Math 231 in the spring semester). Of course, a sample size of 12 students is not large enough to allow me to categorically draw any conclusions, and as such this capstone is better understood as a pilot study which provides insight and direction for future studies. Conducting 12 interviews was feasible under the time constraints of this project, and also provided a fairly diverse range of responses.

Interviewees were identified via snowball sampling. I leveraged the relationships I built in my role as a Peer Tutor for Math 225 (in F18, S19, and F19), and as a mathematics major and prior student in Math 230 (F16) and Math 225 (S17). Using the recruitment email in Appendix B, I reached out to my past Math 225 students and to my peers, many of whom helped spread the word. In particular, a senior math major and friend of mine who was serving as an Undergraduate Learning Assistant for Math 231 connected me to students in that course.

In the recruitment process, I tried to select students with varying levels of mathematical maturity and commitment to pure mathematics, which is a more complete reflection of students' collective experiences. To achieve this more diverse group, I collected demographic data of student's class year, major, and recency of enrollment, and intentionally selected: at least 2 third or fourth-year students majoring in math; at lest 2 third or fourth-year students not majoring in math; at least 4 first or second-years who were taking or had taken the listed courses within the past 6 months. I did not make any effort to compare these different groups of students; I just wanted students across class years and majors to be represented.

I conducted one pilot interview in January of 2020, which led me to adapt the interview protocol to the semi-structured script provided in Appendix A. For the other 11 interviews, conducted in February and March 2020, I adhered to this interview format and kept them to roughly 30-45 minutes in length. All interviews were conducted in-person on Yale's campus, the majority inside of a Bass Library group study room. At the start of each interview, I asked the student for their consent to participate in the interview and to be recorded, following the consent form provided in Appendix C. I recorded interviews using the Recorder app for Android, and the audio and any identifying information were stored in a password-protected folder on my computer.

I transcribed the interviews both using the transcription service Otter.ai and by hand. I opted for intelligent verbatim transcriptions (omitting filler words and phrases like "um," "like," "you know," "kind of," etc.) to improve the readability of student quotes. Following this, I qualitatively coded all interviews in NVivo. I am acutely aware that this subject matter is very personal and close to me, which can inject much subjectivity into my analysis. The process of rigorously coding interviews perhaps helped to mitigate some of this bias.

Literature Review

The aims of this capstone project are primarily of interest to people with direct involvement in advanced university mathematics, such as math professors or students studying math or math-adjacent fields. However, in order to make this project more accessible, background knowledge of proofs and university mathematics (that should be familiar to that group of people) will not be entirely assumed. As such, the literature review first provides context for how proofs are used and taught by mathematicians, and then moves on to review existing practices in proof pedagogy, research studies that document the challenges that students encounter in the learning process, and theories on inductive and deductive reasoning. Of note, this capstone project bridges proof pedagogy research (at the tertiary level) and inductive learning theory, but the intersection of these two areas remains unexplored in the literature. Therefore, this literature review synthesizes research from these separate areas, particularly the difficulties that students have with proofs and the researchers' suggested interventions, in order to demonstrate both that there is room for and that it is necessary to use pedagogical practices that are considerate of the inductive reasoning used in the proof-writing learning process.

Context: Who learns proofs, and why?

Proofs form the bedrock necessary for a deep understanding of mathematics. A proof is "a formal and logical line of reasoning that begins with a set of axioms and moves through logical steps to a conclusion;"[17] in other words, it is an inferential argument that creates mathematical certainty and confirms a mathematical truth. Proofs also allow mathematicians to understand the *why*: in writing (or when reviewing) the sequence of deductions that form a proof, one comes to understand exactly why the correctness of some theorem or mathematical statement can be ascertained from a given set of assumptions. Proofs and mathematics are deeply intertwined. In fact, in a study of mathematicians' perspectives on proofs, researchers found that many considered proofs and proving to be "the very idea of doing mathematics."[18]

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[17] Phillip A. Griffiths, "Mathematics at the Turn of the Millennium," *The American Mathematical Monthly* 107, no. 1 (January 1, 2000): 1–14, https://doi.org/10.1080/00029890.2000.12005154..

[18] Kirsti Hemmi and Clas Löfwall, "Why Do We Need Proof," in *Proceedings of the Sixth Congress of the European Society for Research in Mathematics Education* (Lyon France, 2010), 10, http://ife.ens-lyon.fr/publications/edition-electronique/cerme6/wg2-03-hemmi-lofwall.pdf.

This notion of mathematics as a fundamentally proof-based discipline differs from our more conventional perceptions of math—we think fractions, triangles, $y = mx + b$. This portrayal dominates popular imagination, perhaps because K-12 mathematics is taught largely in a concept or computation-centric way in the United States, where “understanding” means knowing a rule and how to apply it. At the university level, however, proofs begin to emerge in courses like Linear Algebra or Real Analysis, and come to dominate the discourse in all advanced mathematics. It is here that knowing how to apply a rule no longer suffices; “understanding” comes with knowing *why* that rule is true.

For the significant minority of university students who will engage with proofs during their college coursework (i.e., those in disciplines like mathematics, applied mathematics, computer science, physics, statistics, and economics), the contrast between high school and proof-based mathematics is stark. It is worth nothing that the gap exists due in part to younger students lacking the experience and mathematical maturity to engage with proofs seriously,[19] and in part to the insufficient training and knowledge of high school math educators to teach proofs.[20] Whether or not this gap *should* exist is a question outside the realm of this Capstone, but it certainly implies that there is effort required in transitioning students to proof-based mathematics. And, given how different and deeply fundamental proofs are, one might expect that the math community would place enormous emphasis on teaching proofs, and teaching them in the best way possible. But this is not quite the case.

How are proofs taught?

There are two primary ways that students are taught proofs in college: [21] (1) A linear algebra or introductory analysis course is the first proof-based mathematics course the student takes. For the first time and very abruptly, lectures and homework emphasize rigorous proofs over computation and basic understanding. Students are introduced to proofs but learn their

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[19] Yi-Yin Ko, “Mathematics Teachers’ Conceptions of Proof: Implications for Educational Research,” *International Journal of Science and Mathematics Education* 8, no. 6 (December 1, 2010): 1109–29, https://doi.org/10.1007/s10763-010-9235-2.
[20] Tabitha T. Y. Mingus and Richard M. Grassl, “Preservice Teacher Beliefs About Proofs,” *School Science and Mathematics* 99, no. 8 (1999): 438–44, https://doi.org/10.1111/j.1949-8594.1999.tb17506.x.
[21] Robert C. Moore, “Making the Transition to Formal Proof,” *Educational Studies in Mathematics* 27, no. 3 (October 1994): 249–66, https://doi.org/10.1007/BF01273731.

structure by practicing on homework and seeing examples of proofs when theorems and results are proven during lecture; and (2) Marty's Method: Between calculus/computational linear algebra and higher mathematics coursework, students take a "transitional" or "bridge" course that teaches the techniques of proof, usually covering logical operators, induction, equivalence relations, etc., and which may or may not be grounded in a specific subfield of math.[22]

There is strong consensus that the first method is less effective in creating understanding than the second method.[23] Notably, R. H. Marty (the first professor to implement a transition course in 1986) found that students who enrolled in his transition course were 2-3 times more likely to pass a future real analysis class and 4 times more likely to continue pursuing advanced mathematics.[24] However, a cursory glance of course websites of the introductory proof-based math classes of Yale and its peer institutions reveals that the transitional bridge format is not used over the other approach.[25] Yale (like Stanford, Princeton, and MIT, among others) takes the "standard" approach, exposing students to proofs for the first time through Math 225, a proof-based linear algebra course, and Math 230/231, an intensive vector calculus and real analysis course for first-years.

The reasons for this are not entirely clear, but some speculate that introducing a transition course can be logistically difficult for a university, and that students might elect not to take such a course if doing so would interrupt the normal progression along the standard coursework track.[26] For purposes here, the two Yale courses listed above and other non-transition courses are the focus of this project. Typically, proofs are used in these courses in the following ways:

- During biweekly lectures, when a theorem or lemma is discussed, it is often justified with a proof (which usually takes much longer than the theorem statement or examples);
- Students are given weekly homework sets. Problems are mostly or all proof-based in nature (i.e. asking the student to Prove that…), and any computational problems require rigorous justification;

[22] Weber, "Research Sampler 8: Students' Difficulties with Proof."
[23] Moore, "Making the Transition to Formal Proof."
[24] Roger H. Marty, "Getting to Eureka! Higher Order Reasoning in Math," *College Teaching* 39, no. 1 (1991): 3–6.
[25] Institutions without transition courses: Yale; Stanford (Math 113, Linear Algebra); MIT (a 2-week long transition course is only offered during IAP); Princeton (the only transition course, MAT214, is "not recommended for prospective math majors or applied majors."). Institutions with transition courses: Harvard (Math 101, Sets, Groups, Topology); University of California, Berkeley (Math 74); University of Chicago (Math 159)
[26] Weber, "Research Sampler 8: Students' Difficulties with Proof."

- Exams are a combination of computational and proof-based questions. Proof-based questions on the exam are usually designed to take less time to solve than problems on the homework, and therefore tend to be easier.

What is challenging about proofs?

There is overwhelming consensus within the literature that students have difficulty with proofs—in identifying them,[27] in comprehending them,[28] in writing them,[29] and more, with many of these issues persisting under both teaching styles detailed in the previous section. Much more research has concentrated on the former two issues, but the last one—how students struggle when constructing proofs—is the one relevant to this capstone, and hence the one that this section focuses on. To ground a synthesis of existing research, this section will borrow a framework developed by Moore (University of Georgia, 1994) for understanding "major sources of students' difficulties in doing proofs," which is a seminal and oft-cited text within the field.

Through analysis of yearlong nonparticipant observations of an introductory Group Theory course and a transitional proof course, Moore identifies 7 major sources of difficulty that students face when writing proofs, grouped into 5 subcategories:

Definitions

D1. Does not know or cannot state the definitions.

Conceptual (Image)

D2. Lack intuitive understanding of the concepts.
D3. Cannot use concept images to write a proof.

Usage

D4. Fail to generate and use their own examples.
D5. Do not know how to use definitions to obtain the overall structure of proofs.

Mathematical Language and Notation

D6. Cannot understand and use mathematical language and notation.

Getting Started

D7. Do not know how to begin proofs.

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[27] Keith Weber, "Mathematics Majors' Perceptions of Conviction, Validity, and Proof," *Mathematical Thinking and Learning* 12, no. 4 (October 4, 2010): 306–36, https://doi.org/10.1080/10986065.2010.495468.
[28] Alcock et al., "Investigating and Improving Undergraduate Proof Comprehension"; Keith Weber and Juan Pablo Mejia-Ramos, "Effective but Underused Strategies for Proof Comprehension" (North American Chapter of the International Group for the Psychology of Mathematics Education, 2013), https://eric.ed.gov/?id=ED584498.
[29] Moore, "Making the Transition to Formal Proof."

The 7 issues are mostly self-explanatory, and each will be discussed starting from the end:

Difficulty 7, of students not knowing where to start, is actually a point addressed in a variety of proof textbooks,[30] and most educators are aware of this issue. In his concept-understanding scheme, Moore finds that this issue usually results from Difficulties 1-5, which can collectively be referred to as *ineffective proof strategies*, borrowing from Weber. Understanding concepts and knowing definitions sufficiently well to form arguments usually provides a springboard for starting a proof, and therefore D1-5 creates a feeling of D7.

D6 is related to issues with mathematical notation and formal language, which has been studied frequently. For simplicity and clarity, mathematics has adopted symbols (such as ∀ for "for all") and its own conventions for expressing ideas. For example, in the phrase "let vector *v* be in a vector space *V*," the word *let* takes on a meaning similar to "consider any arbitrary (*v*)" or "allow (*v* to be any arbitrary vector)," in order to later on make general statements about the entire vector space. Standard proof techniques, like ε-δ convergence or set equality through mutual containment, use very specific and unfamiliar logic. Studies have shown that students often have a poor understanding of formal notation, which is a problem because formal notation is merely a representation of the fundamental logic of a statement. In a transition course at Tennessee Technological University, students were unable to correctly unpack an informal mathematical statement over 90% of the time (i.e. translate "There is a function *g* such that $g'=f$ whenever *f* is continuous at each *x*" to a formal statement "$\forall f$ [($\forall x$, *f* is continuous at *x*) $\Rightarrow$ ($\exists$ g $g' = f$)]."). [31] They were then unable to understand or use such formal notation in their own proof constructions. To remedy this, the researchers recommended using both formal and informal language to introduce concepts, and clearly delineate between formal proofs and supplementary comments. Other studies related to comprehension have demonstrated student difficulty associated with understanding complex logical statements with multiple quantifiers or predicates.[32]

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[30] Daniel J Velleman, *How to Prove It: A Structured Approach*, 2nd ed. (Cambridge: Cambridge University Press, 1994), http://users.metu.edu.tr/serge/courses/111-2011/textbook-math111.pdf.
[31] John Selden and Annie Selden, "Unpacking the Logic of Mathematical Statements," *Educational Studies in Mathematics* 29, no. 2 (September 1995): 123–51, https://doi.org/10.1007/BF01274210.
[32] Ed Dubinsky, Flor Elterman, and Cathy Gong, "The Student's Construction of Quantification," *For the Learning of Mathematics* 8, no. 2 (1988): 44–51, www.jstor.org/stable/40247924; Márcia Maria Fusaro Pinto and David Tall, "Student Constructions of Formal Theory: Giving and Extracting Meaning," in *Proceedings of the 23rd Conference*

There is also convention surrounding behavior; D6 is tangentially related to the idea of *sociomathematical norms*, that environment and context influences conventional behavior—what can be taken as "mathematically trivial." This term was coined by Yackel and Cobb to describe how students struggle to know what needs to be proved and what can be assumed. For instance, the uniqueness of the identity (that there is only one 1) is colloquially a given. However, it is necessary to prove when students are learning about field axioms or algebraic structures, or on sets where this property does not hold.[33]

The other difficulties that students may face are more related to problem-solving and strategy. D5 involves the structuring of the proof, which can otherwise be thought of as the "flow of logic." Many times, the process of solving a problem and the best sequence of steps for constructing a proof go in opposite directions. A University of Birmingham professor cautions his students: "Although your argument should start at the beginning and then lead to the final statement, while constructing the proof you may want to look at the conclusion and imagine how it may be arrived from the hypothesis. You may then be able to reverse the steps to produce a good proof."[34] The issue goes hand-in-hand with D7, as the starting point has the greatest influence on the logical flow of the rest of the proof, and many studies refer to both as difficulty in general structuring.[35]

D4 and D3 have limited representation in the rest of the literature, but D2 (conceptual understanding difficulty) and D1 (not knowing definitions) are represented in some form in almost every study.[36] In Weber's MAA sampler, he illustrates the hierarchy from D1 to D2:

> "Even if students are logically capable—that is, they know what constitutes a proof and they can reason deductively, recite and manipulate definitions, and draw valid inferences—this does not guarantee that they can construct anything beyond very trivial

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of PME, vol. 3 (Haifa, Israel, n.d.), 281–88, http://citeseerx.ist.psu.edu/viewdoc/download?doi=10.1.1.480.1015&rep=rep1&type=pdf.

[33] Erna Yackel and Paul Cobb, "Sociomathematical Norms, Argumentation, and Autonomy in Mathematics," *Journal for Research in Mathematics Education* 27, no. 4 (1996): 458–77, https://doi.org/10.2307/749877.

[34] Agata Stefanowicz, "Proofs and Mathematical Reasoning" (Universtiy of Birmingham Mathematics Support Centre, September 2014), https://www.birmingham.ac.uk/Documents/college-eps/college/stem/Student-Summer-Education-Internships/Proof-and-Reasoning.pdf.

[35] Selden and Selden, "Overcoming Students' Difficulties in Learning to Understand and Construct Proofs."

[36] Stefanowicz, "Proofs and Mathematical Reasoning."

> proofs. Knowing logical rules and the definition of a concept does not ensure that students can reason about that concept."[37]

D1 can also be further nuanced—aside from not knowing a definition, students may also struggle if they do not know which definition to use. In an abstract algebra course at Rutgers University, even when students were provided with all necessary theorems and definitions to construct a proof about group homomorphisms, they were unable to do so nearly 70% of the time. The author states that "their strategies for constructing proofs were ineffective and crude," [38] and recommends potentially teaching group isomorphisms with more examples to first develop a relational understanding of the concept before providing a formal definition.

The original 7 issues that Moore presented are partially represented in the literature, and can be simplified to four categories: D1 (definitions), D2 (conceptual reasoning), D5/7 (structure), and D6 (language). Collectively, these issues render students unable to communicate mathematical ideas in a logical and coherent manner. Most of the literature does not provide substantial solutions to the problems they identify (for all the studies highlighted, any suggested intervention the author mentioned was described). Many conclude that the professor plays a critical role in shaping a student's learning, and even ones with concrete recommendations tend to concentrate on changes that can be made to the course lectures, which is only one factor in the many that contribute to student learning. This suggests that a different framework—of understanding how students learn to write proofs inductively—could provide new insight on the challenges and solutions to teaching proofs.

Viewed from the perspective of deductive and inductive reasoning (the "drawing of a generalized conclusion from particular instances"[39]), new light is shed on the difficulties identified previously. There exists an analogy between writing English (i.e. writing narratives, prose, etc.) and writing mathematical proofs, which likens structure/style and grammar to proof schemes and mathematical language. Then, D1 (definitions) and D2 (conceptual reasoning)

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[37] Weber, "Research Sampler 8: Students' Difficulties with Proof."
[38] Weber, "Student Difficulty in Constructing Proofs: The Need for Strategic Knowledge"; Keith Weber, "The Role of Instrumental and Relational Understanding in Proofs About Group Isomorphisms," in *Proceedings of the 2nd International Conference for the Teaching of Mathematics* (Hersonisoss, Greece, 2002), 10, https://www.researchgate.net/profile/Keith_Weber2/publication/228857842_The_role_of_instrumental_and_relational_understanding_in_proofs_about_group_isomorphisms/links/557d9bd708ae26eada8db671.pdf.
[39] Martin A. Simon, "Beyond Inductive and Deductive Reasoning: The Search for a Sense of Knowing," *Educational Studies in Mathematics* 30, no. 2 (March 1996): 197–210, https://doi.org/10.1007/BF00302630.

benefit from explicit instruction, much in the way of grammar and vocabulary when learning a new language.[40] This is reasonable, because mastery of theorem definitions or conceptual ideas requires a strong foundation. On the other hand, D5/7 (structure) and D6 (language) are strong candidates for inductive learning, just like writing style.[41] Proof-writing then becomes, fundamentally, a form of writing with a different set of rules. Under this model, different approaches to teaching are more appropriate for different aspects of proof-writing, which further implies the necessity of investigating the exact role of inductive learning and using this information to create interventions for a more effective classroom.

[40] Krashen, *Principles and Practice in Second Language Acquisition*.
[41] Stephen D. Krashen, "We Learn to Write by Reading, but Writing Can Make You Smarter," *Ilha do Desterro 29,* (1993): 27-38, https://periodicos.ufsc.br/index.php/desterro/article/viewFile/8721/8101..

Findings and Analysis

To reiterate the research questions presented in the Introduction, this capstone aims to (1) illustrate how students learn to write proofs and (2) understand the impact and consequences of this approach. The findings in this section are organized in a similar manner. The first subsection examines learning across different proof components and aspects of the classroom, and establishes the crucial role that inductive learning plays when students learn to write proofs. The second subsection evaluates how this method of learning impacts student behavior and the greater course culture. Finally, with insights from this analysis, I propose a set of recommendations to instructors of introductory proof-based math courses to better support the inductive learning process, with the hope that it may also improve students' overall experiences.

Demographics of Students Interviewed

Student backgrounds are summarized below to provide additional important context:

	general information				*proof experience*				*high school math background*			
	Class of 2020 or 2021	Majoring in mathematics	Took Math 225/230 in the last 6 months:	Prior proof experience	Math contest self study	Proof-based course at HS	Summer programs	Other college coursework	Calculus AB	Calculus BC	MVC or DiffEq	Advanced electives at HS
0	●	●		●	●	●			●	●	●	●
1		●	●						●			
2		●	●	●			●		●	●		
3									●	●	●	
4			●						●			
5	●								●	●	●	
6		●	●						●	●		
7		●	●						●	●		
8	●	●		●	●		●		●	●	●	
9		●	●	●	●	●		●	●	●	●	●
10	●		●	●				●	●	●		
11		●	●	●	●				●	●		

Notes: Proof experience is highlighted in blue, and high school math background in green; the gradient coloring indicates a progression of preparedness, which generally translates to greater privilege and access to resources. Interview 0 is the pilot, hence the numbering from 0-11. *Math contest self-study* entails studying old contests and content on Art of Problem Solving, for competitions like the AMC, USAMO, ARML, etc. *Other college coursework* refers to Yale courses in other departments that use proofs (for example, CS 202). *MVC or DiffEq* includes courses taken through local colleges. *Advanced HS electives* refers to courses like Analysis, Graph Theory, etc., that were offered directly by the student's high school (and are uncommon in most American high schools).

I intentionally selected students to be diverse across class year, major, and mathematical maturity (defined as how recently they had taken Math 225 or 230). I was pleasantly surprised to find that the students I interviewed also happened to be diverse in their experience with proofs prior to their first proof-based math course at Yale. Perhaps unexpectedly, exactly half of the students interviewed had already interacted with proofs in a nontrivial manner.

In total, the 12 students interviewed represent 4 class years, 4 different semesters of Math 225, 2 different semesters of Math 230, and 5 different math professors. 10 of the 12 are pursuing or intending to pursue math (including joint with math) and math-adjacent majors; only 3 students are majoring in pure mathematics.

Part 1: Learning to Write Proofs

Writing proofs is a difficult and complex skill, and this subsection attempts to understand the exact nature of the learning process by partitioning it along

1. the content (*How are different proof skills learned?*) and
2. the course design (*How do different aspects of the course, such as lecture, homework, office hours, etc., facilitate learning?*),

and then drawing conclusions about the roles of inductive and deductive learning. The first step of this analysis anatomizes proofs into their smaller components, in order to uncover the essence of proof-writing from students' perspectives and discuss learning on a skills basis. Next, I move on to the classroom, and flesh out how direct instruction does or does not benefit students' learning. Building off themes developed in these two parts, I then build a more generalized picture of how students use inductive and deductive learning when developing their proof-writing skills. Finally, I examine the importance of homework in this process, specifically how different approaches to completing homework aid or hinder it. As noted previously, half of the students had exposure to proofs before Math 225 or Math 230, and so insights from these prior experiences are included as well when relevant.

Proof-writing Components

In the aggregate, students articulated a very coherent framework for the different aspects of proof-writing that need to be learned, with at least four students identifying each of the following components:

1. Mathematical language and notation [0, 1, 2, 5, 7, 9] and basic set theory [4, 5, 7, 10]
2. Using and knowing definitions and theorems [2, 3, 4, 5, 6, 7, 8, 10]

3. Methods of proof (direct proof, proof by contradiction, induction, construction, counterexample, etc.) [0, 2, 3, 4, 5, 7, 9, 10]
4. Choosing how to approach or structure proofs [2, 4, 5, 8], including intuition [3, 5, 8]
5. Conceptual reasoning [4, 5, 7, 8]
6. Sociomathematical norms [1, 7, 8, 9]

Interestingly, these components match very closely with the model developed in the Literature Review, based on Moore's model (of student difficulties when writing proofs) and other related studies. In addition, multiple students [2, 3, 4, 8, 9] named mastery of the fourth component, of approaching a problem, to be the greatest challenge when learning to write proofs. This corresponds with D7 in Moore's concept-understanding scheme, which happens to be the category with the most incoming arrows. As a reminder, incoming arrows indicated that difficulty in the source area could lead to further difficulties in the sink area, so in that sense Moore's model implies that mastering D7 requires mastery of all the incoming D#'s, and hence D7 presents students with their greatest challenge.

These parallels lend support to the validity of the students' framework. But even more, they suggest that at least in the context of proofs and proof-writing, student intuitions can be (and in this case actually are) closely aligned with rigorous observational research. Student voices would be an insightful and valuable addition to these observation-based analyses, and they are the basis for my research.

For most students, mathematical language and notation was "explicitly taught" or "explained in class" [7]. This component corresponds directly with D6 in Moore's model, and a number of students also named "basic set theory" to describe the same ideas of using symbols, formal logic, etc. [10]. Students understood these to be the foundation of mathematical formalism and rigor, and generally agreed that the instructor used mathematical language and notation during lectures in a helpful manner. Said one student: "she would replace English with the symbols, and she would say, 'This is that'…so that was pretty fair to learn" [2].

Knowledge of definitions and theorems, which corresponds directly with D1 from Moore's model, was similarly "learned in class" during lectures [4]. Students also expressed how the same content could be found in the textbook, and those who preferred to study the text learned or internalized definitions and theorems by reading [3]. Proficiency with this component enables students to deconstruct a problem and consider more possibilities for the solution.

Methods of proof were also taught in lecture, but insufficiently so. These standard techniques are important because they provide a basic anchoring and universally understood structure for communicating ideas. Yet multiple students recalled "spending a single day in class on what a direct [proof is], what a proof by contradiction is, or what a proof by induction is," and "only [getting] two or three examples" of each technique [4, 0]. This was helpful and perhaps necessary as an introduction, but ultimately insufficient for mastery of the skill. Instead, for some students, it was the repeated application of certain methods during proofs of theorems in class that solidified their understanding:

> *[For] proving directly, I have a better intuition because that's the main way teachers proved it in class; most of the statements are proven directly. So, I feel like I got a decent intuition of that just by following classes, class lectures and things like that. [4]*

For other students, actually employing the method for a proof on the homework is what helped them learn how to use it, particularly for the technique of induction. As one student put it, "I think I have been introduced to the concept of induction, but it didn't actually click and make sense until after I had used it on a pset" [7]. But students were quick to point out that more difficult than knowing how to use a technique is knowing when to use it, because the wrong technique can lead nowhere, while the right technique can simplify or reduce the problem [4].

This idea is reflected in the fourth component, which groups together sentiments about approaching or structuring a proof (and, again, matches D7 and to an extent D5 from Moore's model). To put this in more concrete terms, multiple students described their view of proofs as a matching problem, of using a string of logic to transition or "match" the givens and known properties to the desired conclusion. Difficulty arises when "there is not always one singular approach to get them to match each other," and students have to rely on their intuition, creativity, and experience to find the right approach [2, 3]. For the most part, students did not feel that they were taught in class how to approach or determine the right approach to writing proofs. For example, one student describes how her instructors would ignore explaining this completely:

> *With regards to how to start proofs, that I feel like was not explicitly taught and more so learned through doing a lot of similar types of problems. I think when sitting in lecture at Yale, a lot of the professors that come to mind, they tend to start the problem off in a very tractable way to finish it, and that leap from the problem statement to where they start is, I think, not really explained very well. So, I had to learn that by just doing things. [8]*

Of course, depending on the instructor, students' experiences can differ. One commended their Math 230 professor for doing "a really good job" of explaining his thought process when approaching proofs and how to generalize this, but also acknowledged that this is atypical; "Usually, not all professors are as good at it" [9].

Still, most students turned to other resources for support. Some viewed reading proofs from the textbook or online, as crucial in developing intuition about approaching problems:

> *In terms of how to start, that I learned from seeing similar proofs online and seeing how this problem might be similar to other ones, because when you see a whole bunch of them, you kind of get a sense of what you might want to do for a specific problem. [6]*

Many credited the interactive aspects of TF or peer tutor office hours, which often involved collaborating with their peers, in growing their understanding of the reasoning behind certain approaches or structures. Students described frequently finding that there would be "a few holes" in their argument or "a jump I didn't get," and turning to TAs or their peers for additional hints or insights helped them both overcome the immediate challenge and apply the reasoning in future situations [4, 9].

The fewest number of students identified conceptual reasoning as a component of writing proofs, and they also disagreed about its importance. Comparing it to the former component (of approaching a problem), one student viewed conceptual reasoning as a trivial barrier:

> *Sometimes I just don't know enough about the material (but I think I usually do have a pretty good grasp on it) and it's silly as that, but sometimes the end proof makes use of some jumps that I was not prepared to make or that I would feel like I would not have come to on my own. [9]*

Another viewed it as fundamental to developing any insight or approach, which is more in line with D6 in Moore's model, about developing and using concept images to represent ideas:

> *Trying to ground something that was pretty abstract in some sort of physical intuition or some sort of imagery in your head, some visualization…the conceptual things were necessary for the insights that you might need to solve proofs. [5]*

In both cases, students explained their learning as a combination of internalizing lessons from lecture during self-study and practicing on the homework.

Finally, students viewed understanding sociomathematical norms as an ongoing point of difficulty, largely because this was something never addressed in any of their courses. Students expressed this concept as not "knowing if what I said was enough" and struggling to "learn what things I could assume and what things I couldn't assume" [7]. They indicated that their ability to answer these questions has increased over time and with more practice (some more than others), but that clearer instructor expectations could have reduced the uncertainty they experienced in the beginning.

Teaching in the Classroom

One of the key themes emerging from this detailed by-parts analysis is that classroom instruction is often not the primary source of learning, especially when it comes to the more challenging aspects of writing proofs. For instance, when students explained how they learned to employ different proof techniques or use advanced conceptual reasoning, they emphasized self-study and practice on homework over learning during lectures. And in fact, when asked to reflect directly about their experiences in the classroom, most students rejected the idea that their professor's teaching played a large role in them learning to write proofs:

> *I think one of the classical answers is like, oh, you learned from a teacher who taught it. But I felt personally that the proofs that we did in class—I didn't think that it helped me learn how to write a proof, because the teachers go through them very fast, and you don't really understand the thought process of everything just given to you. [3]*

Multiple other students echoed that the instructor's pacing when explaining proofs was not conducive to learning, with one lamenting, "there wasn't enough time for me to understand how to write a proof from a one-minute proof that was written on the board" [3]. It is unclear whether this pacing issue is forced by time constraints or an overambitious syllabus, or if this is simply due to the complex and difficult-to-explain nature of proofs. Some students attributed it in part to the disconnect between the mathematical maturity of the instructor and student, explaining how their professor was unable to empathize with students' struggles and therefore did not motivate or explain the proofs at a level suitable for new learners:

> *It seems like professors, sometimes because they have PhDs and they've studied this content for so long, it just seems so trivial and easy to them. And even times, my professors would ask me, "Was the midterm hard?" Even the last question on the midterm was "Was this exam hard? From 1 to 10." I just think they lose track of what is difficult for students who are starting out writing proofs, and to them it might seem like something super trivial and easy. But they lose sight of that. [4]*

For some students, it became necessary to supplement the instruction with "a close study of the textbook" or their notes from lectures, which allowed them to "wrap things together, see where things fit in," and process and internalize the material at a more digestible pace [3, 1]. Not all students viewed the textbook as a valuable resource for learning proofs, partly because different instructors follow textbook content more or less, but those who did felt very strongly about its usefulness as a repository of examples for them to emulate:

> *[The textbook] had proofs in there, and they had very clear definitions, and it was just exactly what was over in class, but with more explanation and motivation that I think I didn't get by going to class*

> *Sometimes, there were similar proofs in the textbook, and I could just kind of model some of my proofs after the ones in the textbook because those were very clear and succinct. At the same time, I wasn't sure if those were too succinct, so I would add in my own understanding of the problem when I actually wrote down things. [7]*

Two students mentioned that their instructor had linked some online proof-writing guides for students to review if needed, but both admitted that they did not take advantage of these resources because they were too overwhelmed [1, 7]. The dearth of useful proof instruction also compelled some students to, again, turn to other resources when completing the homework, like peers, TAs, or the internet. As one student describes,

> *I feel like a lot of the learning how to learn to prove, so to speak, was done outside of the classroom; I don't think it was very well facilitated by the instructor…you had to basically spend a certain amount of time outside of class online, looking at other-people-from-around-the-world's psets, talking to the TF, talking to your friends, scheduling all the time do this. [5]*

Up to now, I have shared a very wide range of student responses, which do not immediately appear to converge in any meaningful way taken on their own. In this next part, I analyze these responses from the perspective of deductive and inductive learning, which organizes these responses in a cohesive and coherent way.

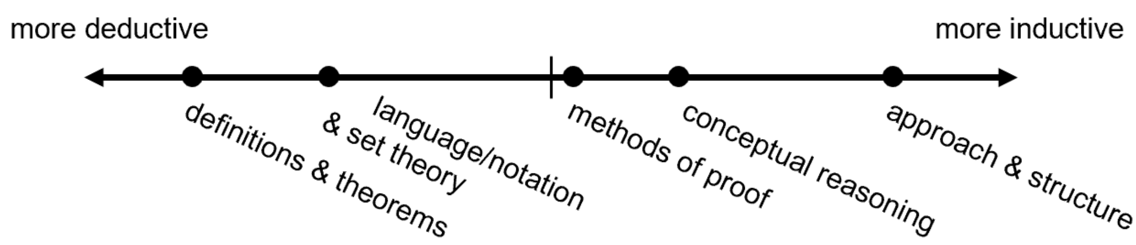
Teaching Proof Components Deductively and Inductively

The many ways in which students learn and are taught different proof components can now be framed using the language of inductive and deductive learning. As explained in the Literature Review, deductive instruction uses a more teacher-centered approach, where the teacher introduces a new concept, explains or demonstrates its usage, and then has students apply it. In contrast, inductive approaches require higher engagement and effort from students, in that students are expected to discover or infer a concept after observing selected examples that the teacher provides.

In the prior two subsubsections, students described certain proof skills as being "learned in class" or "explicitly taught," and other proof skills as "learned from seeing similar proofs online" or "[modeled]…after the ones in the textbook." The former clearly translates to direct instruction (deductive) and the latter inference of concepts via examples (inductive), indicating that instructors in Math 225 and 230 used both deductive and inductive teaching. More precisely, instructors tended to use deductive methods to teach students how to use mathematical language and notation, set theory, and definitions and theorems in proofs—which also happen to be the simpler and more straightforward components of proofs, in that there are rigid definitions or objectively right and wrong ways to use them in mathematical writing.

Methods of proof were taught deductively as well, but only very quickly in "a single day in class." After this, either intentionally or not, instructors switched to a more inductive method by providing examples of direct proofs, proofs by contradiction, etc., through lectures on new theorems and their corresponding proofs. Students seemed to learn conceptual reasoning in a similar way. For learning to approach and structure the problem, however, students sometimes felt that there was little effort on the part of the instructor to even teach this. Instead, they would learn inductively by trying to understand "how this problem [on their homework] might be similar to other ones" and extrapolating. In regards to sociomathematical norms, interviews unfortunately did not provide enough specific evidence for me to draw meaningful conclusions.

To summarize, if we think about the different components as existing along a spectrum of things learned more deductively or inductively, it might look something like this:



This graphic (and in general the component-wise analysis) is useful because it provides a framework and the language necessary to talk about proofs in a clear and specific manner. At the same time, at first glance it can be easy to misinterpret the relatively uniform distribution of components along the spectrum and conclude that students learn to write proofs through an even balance of deductive and inductive methods—that both are equally important in the learning process. This interpretation fails to consider the disproportionate weight that certain components carry relative to others:

In addition to illustrating a distribution along a spectrum, the graphic captures a relationship between the proof components. I noted previously how the leftmost two (*definitions and theorems* and *notation*) are the most content-centric and straightforward, because it is clear exactly what learning each one entails. As we move along the spectrum to the right, towards things learned more inductively, the components become less straightforward and less clearly defined. And crucially, similar to the way Moore used arrows in his model, movement to the right represents a progressive buildup of skills and foundations, where a lack of proficiency in one component impedes success in mastering those to its right. Appropriately, the rightmost component (*choosing an approach*) was the one that students identified as most challenging to learn, as I explained before. This all supports the idea that there is a progression and carryover to learning to write proofs, with components further to the right being the ones that take the most time and effort to master. In essence, components learned inductively are the ones that occupy the most mental bandwidth and dominate students' learning. In this way, learning to write proofs is a primarily inductive process.

Inductive and Deductive Learning: Student Narratives

This claim—that students learn to write proofs primarily using inductive learning—is the central claim of this capstone, which bridges two disparate fields of literature. By breaking proofs down into their component parts, we established how inductively-learned components represent the bulk of the learning. Ultimately, however, the most compelling evidence for this claim (that inductive learning is most important) comes from student instincts. At the beginning of each interview, before delving into the decomposed analysis of proof components and without providing any specific guidance or vocabulary, I asked students to reflect on their learning more broadly. Provided below are five different students' responses to the first interview question, "How do you think you learned how to write a proof?"

> *I think first by **seeing examples of how proofs are written**. And then **find patterns** in ways in which people solve or write proofs, and then start to **pick up those patterns yourself**. It always starts with a predefined approach, and then you start with that approach, and you sort of just write your way through the proof. I think what really helped was **seeing different examples of approaching**, and trying out each of those examples, and which ones would lead you to the answer. [2]*

> *I think it was a lot of learning by trial and error, by using classmates as resources, by googling things, by just trying to **get as much exposure as possible to writing proofs**. That usually just meant **seeing a lot of proofs**…Just like any skill you pick up, you need practice, and the mechanism for practicing proofs, I think, is probably best described as a process of learning common techniques and **seeing them implemented in different settings**—lots of which is calling on these **patterns of logic** when you need to implement them flexibly for some other settings. [5]*

> *I think it's definitely a process of learning a lot along the way. Because once you **see someone formalize their arguments**, you kind of **see different trends** or different tricks they use, or different manipulations of the formula that you could apply later. [4]*

> *I learned proofs by doing proofs and reading proofs online, for certain math problems, the contest problems that I was studying…And then as I took more math classes and I was **exposed to more and more problems**, I realized that there were **certain patterns of proofs** that were associated with different kinds of problems. And through going to lecture and reading textbooks and online resources, I found that I just had a better idea of where to start and where to formulate the problem to go from there. [8]*

> *The first time I wrote proofs, like I said before, was in preparation for contests. It was for the USAJMO, and the way I learned was by doing practice problems and **seeing how the solutions on AoPS—how they structured their proofs**, and **copying**, in a sense. [0]*

Clearly, these students use language that points us towards inductive learning. In the underlined phrases, students repeatedly share how they learned by first *seeing examples* and *gaining exposure* to different proofs and problems, and then *finding the patterns* in what they saw. This matches extremely closely to the definition of inductive learning used in this project: discovering concepts and rules by observing examples. Again, this question was posed at the very start of the interview, so students were not primed with my definition in any way, which makes the consistency of their responses even more significant. In students' minds, even though they lacked the vocabulary to identify their learning method by name at that time, writing proofs was very much something they learned to do inductively.

The natural next question is whether students would give the same response after being provided with this vocabulary of inductive learning. In designing the interview, I actually did want students to have the opportunity to reflect on their learning from the same inductive/deductive perspective that I held—to really tackle the core research questions head on. I had a strong suspicion that many students' intuitions would lead them there (as was just demonstrated in the paragraphs before), but I was also unsure whether *all* students would be able to reach that point on their own. So, during the middle of each interview, I defined the terms d*eductive learning* and *inductive learning* and explicitly asked: "Do you think you learned how to write proofs inductively or deductively, or both, or neither? And how so?" Student responses to this question remained consistent with those initial reflections. In fact, *all* eleven of the students who were asked this question agreed that their learning of proof-writing skills was primarily inductive. Students would adapt the definitions provided to the context of their math course, and explain their learning as:

> *Probably more inductive, by just observing how the book does it and how the teacher does it, applying the similar skills to a completely unknown problem, like on a test. I think that was pretty inductive. [3]*

The sentiments expressed and specific phrasing are very much aligned with the initial reflections above. Some students were also able to distinguish between different components of proofs, based on the discussion from earlier questions. For example, this student acknowledged that their learning did include some deductive aspects:

> *I think some of them I learned deductively, like notation and "single arbitrary x," but basically everything else I learned inductively through office hours, TA sessions, whatever—him walking us through this problem and then me realizing, "Oh, so that's an actual proof technique." [7]*

Still, the student's attitude suggests that outside of *mathematical language and notation*, the more effortful and bulk of proof-writing is learned inductively.

Overall, the consensus of students' narratives in this cohort of interviews suggests that learning to write proofs is an inductive process at its core. As student quotes have illustrated, the "examples" that students use to infer and develop larger skills sometimes come from proofs during lecture, and other times from proofs in the textbook; both were discussed at length in the subsubsection about teaching and the classroom. The part of the learning process (and aspect of these math courses) that has not yet been addressed is *homework*. In this final part, I will explain how homework and inductive learning are inextricably linked.

The Crucial Role of Homework

During interviews, when explaining how they learned to write proofs, every single student brought up “homework” before I had the chance to prompt them. Especially given the shortcomings of direct classroom instruction experienced by many students, homework (in the form of proof-based problem sets, or psets) served as the most important part of the course and the learning process. As one student dramatically puts it, “If I had not had homework, then I would never have learned” [1]. Students described taking a myriad of approaches to completing proofs on the problem sets, which ultimately boils down (in a fairly intuitive way) to a total of three. In this subsubsection, I examine each approach in turn: why students choose it, what it entails, and how it impacts learning. The 3 approaches that students take are:

1. Deriving the solution and proof to the problem *on their own* (allowing for use of resources which provide content knowledge, such as the textbook or Wikipedia);
2. Deriving the proof *with significant guidance* from a teaching assistant or peer;
3. *Reconstructing* or reproducing the proof *after a solution has been revealed* by a teaching assistant, by an online resource (math.stackexchange.com or old homework solutions from external course websites), etc.

These were actually borrowed from a student’s response, which is much simpler in nature:

> *In that sense, [the approach I take] sort of represents a spectrum, where **doing it on my own** is maybe the easiest sort of problem; **doing it with some help** is maybe medium; and **doing it with 100% help** is maybe too difficult. And I suppose the middle is the right thing to go with, and most helpful for learning, because that gives you the right balance between being able to do enough of the problem for yourself so that you’re not relying just on others, but also getting exposed to things that are challenging to you. [9]*

As he explains, there is a clear natural progression from the first approach to the other two, with each transition spurred by increasing difficulty. In practice, students follow a process like this:

> *I would try to do the problems on my own at first. So, at first, all of it was me, but then whenever I couldn't go further, I would ask peer tutors or peers, or I would look at online resources. So, at the end of the day, probably every problem I checked with a peer tutor or with a peer... But in terms of how much I tried by myself at the beginning, it would be like 0% [external] resources because I at least tried to do all the problems, I think, before I would ask others. [3]*

In addition, the first two approaches are usually encouraged or condoned by instructors of proof-based math courses, and the last one is not.

Approach 1: Deriving the proof on your own

The first approach relies solely on the ingenuity of the student to derive the proof. From understanding and solving the problem statement to constructing a formal proof, the work is the student's own. To be more specific, students may consult resources that provide content knowledge or a review of proofs in general, but these resources should not deliberately disclose the insights necessary to completing the problem. As one student cheekily puts it, the first approach is "doing the homework and making sure that I did [it] and not someone else" [5].

Students generally regarded this approach as the "best" one to take. Having the ability to be self-sufficient and self-reliant demonstrates competence and mastery to students themselves, and therefore inspires positive and confidence-boosting feelings. Compared to the other two approaches, this approach "sparks a greater amount of joy," provides "more personal satisfaction," and grants them a "[greater] degree of ownership" over their proof and the problem [8, 2]. Moreover, staying committed to independently completing the proof teaches the value of "the struggle," the creative process of exploring and exhausting discovered avenues and attempting to find new ones, which is "super important for learning how to problem solve" [7].

That is the greatest benefit of employing the first approach: the student has the opportunity to practice and become more confident in proof-writing skills that they already have some degree of familiarity with. During interviews, some students were surprised when they made this deduction, explaining how in some ways this ran counter to conventional beliefs they held about homework and learning:

> *I guess I traditionally thought, if I can do a problem on my own, that that's the best for my learning. But usually, in those cases, if I can do it entirely on my own, there's not really much of a challenge to it. [9]*

The impact of this first approach on learning is usually not the acquisition or development of new or better proof-writing skills, but rather the reinforcement and refinement of already existing ones. As such, problems that students are able to *solve on their own* are only useful to the extent that they are opportunities to practice.

Approach 2: Deriving the proof with guidance

Introductory proof-based math courses at Yale provide employ two types of teaching assistants (TAs) as additional resources for students who may find them more approachable or relatble than the instructor. There is a graduate Teaching Fellow (TF) who hosts biweekly discussion sections, and one or more Undergraduate Learning Assistants (ULAs) or Peer Tutors who host biweekly office hours.[42] In addition to the instructor's office hours, these sessions are an opportunity for students to review concepts, ask questions, and get help on the homework. The second approach to homework, of deriving the solution with hints from another person, usually takes place within these sessions. For some students, this was the default environment to seek out when they encountered "unapproachable" problems [7]. For others, getting hints from a TA's session was the last resort:

> *There were always one or two problems I had to go to office hours or the peer tutor to find out…Like there was no other way for me to solve that problem…I've looked online, and there's nothing. And I don't know what to do with what the book's doing, and my friends also don't know what's going on. [1]*

To be clear, these TA sessions can be part of both the second and the third approach, depending on the TA's actions. If, for instance, the TA does a complete walkthrough of a problem, this would count as the third approach, and is not discussed here. The second approach really involves a "small group setting" which is highly interactive and allows students to engage in a guiding dialogue with their TA [4]. More specifically, the guidance that some students sought came in the form of immediate feedback on their ideas for solving a problem. This student describes how their TF would solicit possible approaches for tackling different parts of the proof, and then evaluate the feasibility of the approach together with the student:

> *I think [the TF], during his office hours, really led things in a very interactive way. He would be like, "Oh, how would you start this proof?" Or, "How might we do the next part of this proof?" It really involved people, and I think as a result, it made learning proofs a lot better. I think a very interactive process with proofs is good because again, you might think something would work, but a person who really understands proofs would know it doesn't work. And then when you suggest it, then they're like, "No, that wouldn't work; this is why." That gives you an instance where you might not want to apply a certain technique. I think having an interactive experience with somebody who has a lot more experience with proofs is really important. [10]*

Some students preferred more direct guidance, where at the student's request, the TA would nudge the student in a particular direction or give a hint that reveals a substantial piece of insight required for deriving the solution:

> *I'll say, "Hey, so I'm struggling with this problem." And then I'll describe what the problem is and also describe the context, where I'm like, "So I get this and I get this, and I've gotten to this point, and I just don't see how you can use this information to get to the next thing." And then the TA might say something like, "Yeah, I would think about this particular theorem" or "What do we know about this part of the problem? What have we learned about that in previous problem sets or in class or something? And how might we leverage that here?" Something like that, where I'm basically just invited to play with a particular*

[42] For the most part, students were not aware of the distinctions between these titles during interviews. I will do my best to disambiguate when it matters, but in many quotes when students talk about a "TA", they are referring to the graduate TF.

> *thing. And then I do so, and usually then I'm able to figure it out on my own…I think having TAs who just give you the right nudges at the right moments is the most helpful for the learning process that I've had. [9]*

With these two quotes, the differences between the learning involved in the first approach (deriving on your own) and in the second (deriving with guidance) become clear. Left to their own devices, students are hard-pressed to develop new insights, because they can only do and conjure up as much as they know. In conversation with a more experienced mathematician, however, they gain exposure to different, more mature ways of reasoning at the critical points where they are initially stuck. In surmounting this barrier, students may see how a proof technique may be used in a novel way, how to think about certain types of problems, etc., while still preserving the majority of the proof derivation and problem-solving process for themselves. By understanding their TA's intuitions, students can develop theirs:

> *I do think you should struggle over a solution to a certain thing, but simultaneously, a lot of times people who you can go to and ask questions are people who really understand it, and I'm a big fan of learning from how people think, that are really good at something. Because I think it also helps you to think more efficiently, in those ways. So, I really liked utilizing the peer tutor, the TAs, and the professor, because they're all very good at this. And I want to know how they think so I can think similarly, or at least know the ways that they approach the problem so that I can. [10]*

The language in this response strongly echoes our discussion of inductive learning, specifically for the proof component of *choosing an approach*. Through these TA sessions, students are exposed to the thinking and approaches of successful, seasoned TAs—in essence, a set of examples to model. From there, students may attempt to mimic their TAs' reasoning in similar problems, or discern the parts that resonate with them to develop a framework of their own.

Peer collaboration was also a useful process for many students. Peer Tutor sessions frequently enabled collaboration between students, especially among those that may have different amounts of fluency in proof-writing. A student whose study group consisted mostly of students who had taken a proof-based math course already received guidance similar to the kind a TA might offer:

> *Also, I would work with other people who I think had taken proof-based math courses. I would be like, "Yo, what is this? What are you even saying here?" And me arguing with them and learning through this process of arguing and realizing, "Oh, this is what they're doing, and this is why it shows this," right? [7]*

Students who are at different stages on the problem set or who have completed different problems can also offer guidance to their peers. As one student describes,

> *Working with other people, you always see what arguments might not work, and better ways to prove what you're trying to say. [1]*

Of course, peer expertise is not always as dependable as a dedicated TA's, but the shared experience that students have often make them best at communicating with each other. Students are also collectively quite good at problem solving, when they leverage each person's individual insights to craft a coherent solution, which is collaboration at its core:

> *But then in a setting with multiple people, where you're working on something collaboratively, if you have a good idea, and it's good because it's insightful somehow, in terms of it brings something to the problem— And maybe that's not the whole solution, but then the others can work with that, and they'll say, "Yeah, that is a really good idea. What can we do with that?" They'll maybe transform it in the way that's necessary, or I'll see the way it needs to be transformed. A lot of the times, speaking aloud, "What are the assumptions that I'm working with here?" lets me see what the next step is, even if I was going to get stuck on that on my own. [9]*

The learning that occurs in these instances is similar to situations involving a TA, but there is more effort required on the part of the student to distinguish between productive ideas and less relevant ones. Still, exposure to others' ideation and thinking helps develop a student's intuition.

Finally, students in multiple interviews point to the difference between the graduate and undergraduate teaching assistants. Because they have more experience, graduate TFs "know a lot more" and "have a really good understanding of math in general," and so can anchor and explain the problem in a broader context when that is helpful [9]. However, the general consensus (of seven students, across three different graduate TFs) is that their TFs' advanced backgrounds prevented them from understanding students' difficulties and communicating in an effective way. As one student describes,

> *Sometimes I think [my TF] wasn't very good at teaching people. He kind of didn't understand how hard things were, and that made it difficult, because he would always be like, "Oh yeah, this is trivial" or "This is just by definition." And you just had to kind of persuade an explanation out of him, which was very difficult to do. [7]*

Others agreed that the TFs sometimes would "explain the problem in a very convoluted way" or use ideas and approaches that were "a little bit too much for me to be able to understand" [8, 9]. A few students actually had overtly antagonistic experiences, where they felt judged or put down by their TF for not understanding or knowing how to do something. One student actually said:

> *I think that some TAs here, and especially [my TA], I think he's hoping for failure as opposed to success…They don't want to help you; they want to watch you suffer and be like, "Good try." [4]*

In contrast, peer tutors, even though they may not have been as proficient at solving difficult problems, were able to "describe the process in a way…[and] just make suggestions that I can understand more easily" [9]. This is perhaps due to the proximity of their experience to students', which translates to better recall of what it may have felt like to struggle with certain concepts or problems as a beginner may.

<u>Approach 3: Reconstructing proofs after seeing a solution</u>

The third and final approach is the one that I was most interested in understanding. From my experiences as a Peer Tutor and student, I knew this was an approach that students commonly took, but that instructors rarely ever addressed, except to disallow it (probably because they believe it hinders learning, or essentially amounts to copying work or cheating). I wanted to uncover how students felt about this approach and what its impact on learning was.

I first want to establish what the third approach—of reconstructing or reproducing a proof after a solution has been revealed—looks like in practice: There are two main ways students encounter such solutions. First, the TF or Peer Tutors may provide a sketch or detailed walkthrough of the solution during the TA sessions. As one student explains,

> *Basically, it was going to TA office hours and him giving a sketch of the proof, and me taking notes on that, and then thinking about it and understanding why that's true, and then just kind of writing that up. That was the entire process. [7]*

Or, students might look up the problem on the internet. This was usually quite successful, because the fundamental concepts in introductory linear algebra or analysis are quite established, as are the problems which test their understanding. Students reported finding solutions posted on a math-focused discussion forum (multiple students mentioned ProofWiki, mathoverflow.net, and math.stackexchange.com) or "on old psets that some professor had let out into the ether and then filled in solutions for" [5].

After seeing this solution, a student then needs to write a proof for the homework. The language I use to label this behavior is very intentionally *reconstructing* or *reproducing* proofs, and not *copying* or *cheating*. Math 225 and Math 230 have reputations as being very challenging and work intensive; most Yale students who elect to enroll in these courses are highly-motivated to learn the material, and the highly-motivated students I interviewed definitively rejected the idea of copying something mindlessly to the detriment of their learning. They sought out solutions not to cheat on their homework or get by easily, but to support their learning.

All of the students who admitted to taking this approach expressed a high degree of commitment to understanding every part of the solution they saw: the approach, the techniques, the intuition, the logic. After this process, they would rewrite everything in their own words and add in their own explanations to reinforce their understanding. As one student says:

> *I've never been somebody who copies somebody else's work on the internet. And even now, I still do similar things: if I see a problem and I look at it for 30 minutes and I don't know how to solve it, I'll look up how someone solves it. But I'd say my psets still take me 10 to 12 hours to do, because I truly try to understand every step of the proof and understand why this statement holds…I don't think I would do anything that I would consider cheating. And I think that, like I said, the way I formalize my arguments is that I looked online, I found people that wrote these proofs, and I followed them step by step, and I really*

> *did understand them. And by the end…I felt like I understood the basic logic that was used and the way that they got through the argument. And as always, I think I'm always someone who writes more, so when I write my proofs I'd always write what I thought the explanation was, so that's why I didn't feel like I was being academically dishonest and I don't believe I was being academically dishonest. Because although I was very much looking at how someone else constructed their argument, in the end, I would formalize it to myself and write my own intuition in the margins, like why I thought it was true. [4]*

They were also very committed to writing down only what they understood, leaving it blank if this was not the case, or, as one student humorously relays:

> *I don't want to just write down what I read…I'd really try not to write something down unless I understood it. And if I didn't understand it, then sometimes I would just make up a wrong incorrect proof. [laughs] [6]*

Some students used a "delay" approach, where they tried to build their understanding by processing a solution and then rederiving it after a period of time:

> *What I tend to do if I can't solve a problem is I look it up. But then I don't write it then and there; I wait a day and then write it up the next day, attempting to solve it in the same manner as what I had seen. [8]*

And some students also interacted with a solution by treating it as a series of progressive hints:

> *But for Linear I would always try and do it on my own, and if I couldn't get anywhere I'd look it up. Then I would usually try to read a line of what they say and then try it myself. [6]*

This all begs the question: if students are so highly motivated and committed to their learning, why even choose this third approach? Logistically, most students explained it as a combination of reaching a point of stagnation on a problem

> *[Meaning that I tried for a very long time and nothing's happening anymore. Like no matter how much work I put into that problem, I feel like I can't get the answer by myself. [3]]*

and feeling "time pressure" from other commitments, which barred them from spending more time on the problem sets or going to TA sessions [2]. Students also agreed that there is a point in which productive struggling becomes just wasting time, and a balance must be struck:

> *I feel like for me, personally, it's a trade-off between trying to do it all yourself and spending significantly more time, over working with others or looking online, and still getting the same or a similar understanding. [0]*

Crucially, this student describes how he does not sacrifice his understanding of the proof when finding this balance. Indeed, most students strongly believed that taking the third approach (of reconstructing a proof already seen) did not inhibit their learning. As evidence, a few cited their success on the exams in the course, and their continued success in math courses they had taken since Math 225 or 230. In fact, not only does the third approach not inhibit learning, but it also *enhances* students' learning when done right, much in the way that deriving a proof with guidance does:

> *I think I probably learned more from [problems whose proofs I had to reconstruct] than I do from the ones that I solve on my own, just because those are the more challenging ones, and they teach me a new way of looking at problems or making use of things we've already learned. [9]*

This student describes gaining exposure to new ways of reasoning and new approaches, similar to the kind of inductive learning involved with the second approach. The key difference here is that the entire proof is provided, which conforms to the ideal setup for inductive learning. One student recalls how he learned how to approach proofs involving direct sums by closely studying a TA's walkthrough of a homework problem:

> *First off, you do pick up different tactics and ideas about how to tackle something, and what a proof for something more complex might look like. And I think also, it gives you exposure to proofs in a certain area where it makes proving things in that area a lot easier down the road, if you understand— For example, a first proof with a direct sum, you might have no clue what's going on, but once you are kind of walked through with how to prove that, then in the future direct sum proofs are not that bad. [10]*

Moreover, instead of having only hints and key insights to "discover" or infer proof skills from (as with inductive learning in the second approach), students employing the third approach build a repository of complete, complex proof examples through the solutions they find. And because students have dissected these proofs line by line when reproducing them for their homework, students are usually intimately familiar with these proofs and especially attune to new approaches and reasoning. The result of this extensive close study (sometimes even more so than examples from lecture or the text) is effortful, effective inductive learning:

> *I think it is a process that requires a lot of thought. And it isn't taking someone else's work for granted because I think you are really focusing on understanding the logic of their argument and understanding why certain things that they stated are true. And I think it's definitely a process of learning a lot along the way. Because once you see someone formalize their arguments, you kind of see different trends or different tricks they use, or different manipulations of the formula that you could apply later…*
>
> *…But I felt like, more so than even doing it on my own, by looking at someone else's work, who obviously had more experience in it, who really put a lot of thought into it—they've put weeks of thought into it—just reading the way that they did it, I really do feel like I gained a better understanding of proof writing overall and how to formalize proofs. And although I am not a proof expert, I do really feel like, more so than my other classmates who took Math 225, I know how to formalize an argument, and I can really see the connection between different steps in a proof. [4]*

This third approach is a prevalent mode of learning. Nine of the twelve students interviewed used the third approach (the three who did not were the exact subset of students enrolled in a certain course at a certain time, suggesting the culture of this course may be different). A few students expressed how "in a way, everyone does it," [1] and one student spoke very passionately about his frustration at how necessary and ubiquitous the third approach was:

> *To me and everybody I talked to, there's this gaping hole in the Yale Math department, that they expect people to do these ridiculously hard proofs, and I feel like that's just a constant thing. Oftentimes for people, it's not a matter of, "Have you looked it up online?" It's like, "What did you find?" Like, "Who's found the best resource to do it?" [4]*

Of course, students varied in the amount that they each utilized the third approach, but at the more severe end of the spectrum, one student used “maybe half at best” [8] to describe the amount of problems he derived without seeing a solution, and another said:

> *I would have had to drop that class if I couldn’t look stuff up online, because I had no idea how to do half the proofs. [6]*

The third approach is sometimes an act of desperation for students, but it is also incredibly valuable to the inductive learning process and to learning to write proofs.

Part 2: The Impact of the Inductive Approach

In the previous subsection, I illustrated how the inductive approach impacts student learning: how it is used to teach students to write proofs, and how inductive learning is embedded in many aspects of the course, like TA sessions or the homework. But beyond impacting student learning, the teaching method an instructor chooses to employ has profound consequences for the course culture and students’ mindsets as well; this subsection focuses on three. As a result of the inductive approach to teaching proof-writing, many students

1. become highly dependent on online resources (to complete proofs on the homework);
2. engage in active collaboration with their peers; and
3. become proficient at self-study.

Widespread participation in these behaviors can create cultures of collaboration or self-driven learning, as well as a culture of overdependence.

For each of the three behaviors, I describe the emotional response that participation in the behavior elicits (for example, its effect on a student’s confidence, attitudes towards mathematics, or interest in pursuing math further). In addition, I analyze how instructor expectations surrounding each behavior (or students’ perceptions of their expectations) shape this emotional response. To conclude this subsection and broadly understand the impact of the inductive approach, I share students’ opinions and reflections on their own learning, inductive or not.

You may notice that the three behaviors I named look very familiar. And indeed, they were each explored to an extent in the last subsection. But their inclusion there is motivated by the role they play in *learning*: these behaviors facilitate the learning of proofs, and were therefore necessary to include in a discussion which concentrates on understanding the inductive learning process. In contrast, these three behaviors are the focal points of *this* subsection because they are behaviors which emerge in response to the inductive teaching method, whether that is as a natural symbiotic byproduct, to address inherent inadequacies of inductive learning, or to remedy a poor execution of the method. Additionally, whereas I refrained from passing any judgement in the previous subsection, I now seek to evaluate the learning process in the broader context of a student's wellbeing.

The causal relationships between inductive teaching and each behavior is similar in nature for all three behaviors. In an earlier discussion about the crucial role of homework, I established how students rely on their peers and on internet sources (part of approaches 2 and 3) because they do not feel adequately prepared by the classroom instruction to solve and complete proofs on the homework on their own. I also demonstrated how many students turn to self-study (of the textbook, for example) when the course lectures fail to bring them to a point of understanding. It is reasonable to attribute this to poor teaching, but because these experiences are common across multiple sections of introductory proof-based courses at Yale, it is more than likely that the inherent qualities of the embedded inductive teaching style play a large role. During interviews, the repeated expression of sentiments like "I didn't realize I was learning," or that "proofs would come from nowhere," or constantly using "proof concepts that we hadn't learned" suggest that the inductive teaching is not sufficiently explicit to give students a sense of agency and control over their learning [2, 10, 4]. This in turn forces students to pursue other avenues of learning, which manifest as the three behaviors discussed here.

High Dependence on Online Resources

The third approach to homework, discussed previously, involves reconstructing or reproducing a proof after a solution has been revealed, with many students obtaining this solution by searching the homework problem on the internet and finding proofs on math discussion forums or in the materials of an old course website. This behavior is very common, as I established before, and during his interview one student recognized how its prevalence had fundamentally transformed the way he and his peers viewed the homework and the process of writing a proof, identifying the transformed result as "a culture of looking online:"

> *I think it kind of creates a culture of looking online and figuring out how to do it and how to best write it. So, I think probably, in that sense, a lot of my homework assignments I learned through listening to the way other people did it and being receptive and trying to, like, not—like I said, not* "copy"*—but just transfer that to those skills that they used to make an argument. [4]*

Most students were aware of this "culture," and could choose whether or not they wanted to participate in it, but those who were not could suffer academically and mentally. I happened to interview one student who remained oblivious to this "scene" and an outsider to this culture until far into the course. He describes how he spent the majority of his semester trying to perform at the same level as his peers and marveling at how easy things seemed to come to others, only to discover that this comparison was meaningless because many of those students were operating under a completely different set of norms:

> *Well, I think something that was kind of a relief throughout, I realized that a lot of people were just looking answers up. It came up during a peer tutor session—somebody told me that basically the entire class just looked up solutions. And I hadn't been doing that, and I was always wondering why the pset average was always super, super high. And I realized that that was probably because most people just look things up. And there was also a whole scene that I wasn't a part of: finding solutions together, or going into office hours and just getting solutions. People told me that they just got the solutions at office hours or other things like that. It was kind of discouraging, but also made me feel not as dumb because I realized for other people the problems were also not approachable and difficult. [7]*

These two quotes clearly reveal a deep-rooted culture associated with using online resources.

As I established before, students do not regard this behavior as cheating or as academically dishonest, and I also hesitate to assign such labels to this behavior. The *Yale College Undergraduate Regulations* leaves the determination of academically dishonest behavior on homework assignments to instructors, stating that "students are expected to ask instructors for a written explanation of what kinds of collaboration are appropriate;"[43] and the *Handbook for Instructors of Undergraduates in Yale College* suggests to instructors that "improper collaboration on assignments"[44] counts as a form of academic dishonesty. Given the ubiquity of technology in today's society, one would expect instructors to acknowledge and address how students may interact with online resources in their course syllabi. However, of the six different course sections spanned by student interviews, only *one* course's syllabus explicitly mentions the internet (and it instructs students: "But you will not look up solutions online, nor will you directly copy work from others."[45]).

Unsurprisingly, students enrolled in this particular course were the only ones I interviewed who had a clear understanding of their instructor's expectations surrounding the use

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[43] "Academic Dishonesty," Yale College Undergraduate Regulations 2019-2020, (2019), http://catalog.yale.edu/undergraduate-regulations/regulations/academic-dishonesty/.
[44] "Academic Dishonesty," Handbook for Instructors of Undergraduates in Yale College 2019-2020, (2019), http://catalog.yale.edu/handbook-instructors-undergraduates-yale-college/teaching/academic-dishonesty/.
[45] Patrick Devlin, "Course Syllabus," Math 230 Canvas Homepage, (2019), https://yale.instructure.com/courses/50429/assignments/syllabus.

of online resources. For the rest of the students, the majority expressed not knowing whether their behavior would be allowed, expected, or condoned by their instructor:

> *I don't know. I think in a way, everyone does it. Everyone goes to the peer tutor; everyone looks things up. I don't know if that's what the professors want. [1]*

Of course, students are not ignorant, and they tended to evaluate the behavior using an intuitive understanding of what is and is not academically fair (i.e. *not* using online solutions would be more academically honest than using them). Still, students frequently rationalized their use of online solutions by pointing to its positive impact on learning:

> *I think at the beginning of the semester, just as someone who'd never done proofs before, I felt like looking at someone else's work, or perhaps looking at someone else's argument, or the way someone else formulated something, was perhaps, I don't know…I try to shy away from it as much as possible. And I was super afraid to utilize those resources at the beginning.*
>
> *But then towards the end, and I think as the process sort of progressed, I realized that these resources were actually super helpful to me. And I think a lot of times the reason people are so down on looking at these resources is because they think that they inhibit student learning, right? Like, why would you? I think that's the main reason why the Yale office is super down on cheating. And I agree in most cases, like when you're just copying down an answer and turning it in, you're not really understanding it. But I felt like, more so than even doing it on my own, by looking at someone else's work, who obviously had more experience in it, who really put a lot of thought into it—they've put weeks of thought into it—just reading the way that they did it, I really do feel like I gained a better understanding of proof writing overall and how to formalize proofs. [4]*

Rather than feeling guilty or ashamed, this student views utilizing solutions online as necessary and very productive. And while not all students I interviewed were as defensive or strongly opinionated as he was, they took a similarly flexible approach in judging their integrity:

> *I think…unless they explicitly forbid me from looking things up…I don't know if there's an expectation, but I take it as: as long as I am producing my own work and I feel like I'm learning, then I think it is okay. [8]*

Some students also felt they were sent mixed signals by different teaching figures in the course, making worse their confusion surrounding instructor expectations:

> *And even my current peer tutor is like, "Yeah, this is online," like, "I recommend looking it up," even though the syllabus is just like, don't look anything up.[46] So, I just think professors set ridiculously unrealistic expectations with that regard. [4]*

This lack of clear instructor expectations serves to exacerbate the culture of dependence and "looking online." While I do not want to call this cheating, research on academic dishonesty and the proliferation of cheating on college campuses in the internet age is very relevant to this discussion. Donald McCabe, a Business Professor at Rutgers University, has written extensively

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[46] This student actually misremembered the course syllabus, as it makes no such reference to usage of the internet, and is likely just inferring his instructor's expectations based on his own intuitions.

on the subject and conducted a decade-long nationwide survey spanning nearly 70 colleges and 40,000 students. One of his key findings is how strongly contextual factors can shape students' approach to academic integrity, particularly peer cheating behavior and peer disapproval of peer cheating behavior.[47] As he writes, "Regardless of the campus integrity policy, if students see others cheating, and faculty who fail to see it or choose to ignore it, they are likely to conclude that cheating is necessary to remain competitive."[48] This is exactly the situation occurring in Yale's introductory proof-based math courses. The instructor fails to establish expectations about the approach to homework, and students adopt a culture of overdependence on online resources as the norm. And while this can benefit the inductive learning of writing proofs, it has consequences for students' confidence.

Mathematics can often feel like an incredibly results-oriented discipline, where the ability to prove problems is a direct indication of mathematical skill or potential for success:

> *I have this all the time, and I know a lot of people have this all the time...But there are times when I am just distraught by [math], and I feel like I'm just not any good at it. That's usually when I'm getting a string of problems that I just can't figure out, and I need lots of those hints and nudges, and I'm just not seeing how to get to the solution, even if it's what I feel like should be an easy problem (maybe it's not worth a whole lot of points, or something arbitrary like that). Those can be really bad for my confidence in those moments, because I feel like, as I said, "I'm not any good this, I don't know why I'm studying math"—that kind of thing will go through my head. But there are then also moments where I do feel really proud of my abilities, sometimes. Just yesterday, I felt like I had a string of problems where I had lots of really good ideas all of a sudden, and that was really [nice]. [9]*

When students leave classroom lectures feeling that they have not been equipped with the tools or knowledge necessary to complete their homework, at least in part because of the inductive teaching method used, many of them feel forced to resign themselves to a path of exploiting online solutions or other resources for the sake of their grade:

> *I think it just made it more anxiety filled and less fun to not feel like you walked out of class being able to do the homework always. Or that you knew you needed to talk to other people about it, or look it up online, so you needed to now schedule into your week times to talk to other people and work it out with them. [5]*

This student's descriptions of an experience that is "more anxiety filled" and "less fun" begin to paint a picture of the emotional impact of the culture and normalization of relying on online

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[47] Donald L McCabe and Linda Klebe Trevino, "Individual and Contextual Influences on Academic Dishonesty: A Multicampus Investigation," *Research in Higher Education 38,* (1997): 379–396, https://link.springer.com/article/10.1023A:1024954224675.

[48] Donald L McCabe, "It Takes a Village: Academic Dishonesty and Educational Opportunity," *Liberation Education 91*, no. 3 (2005): 26-31, https://www.aacu.org/publications-research/periodicals/it-takes-village-academic-dishonesty-and-educational-opportunity.

solutions. Students reported that “[having] to use the internet so much as a crutch to do the homeworks” made them feel “stupid” and “bad”, and resulted in feelings of “helplessness” or of “not being confident in the material” [1, 6]. And repeated engagement in this kind of behavior can become discouraging and harmful to a student’s confidence not just in the material, but in themselves and their mathematical ability.

> *I think happening once or twice, the effect is very small. But if you think about length of an entire semester, leading up to a final exam or midterm, if you constantly have these situations, where you found yourself being in this situation where you have to rely on external hints to solve problems, it can make you feel very incompetent in confidence (and I’m just using “you” in general)…It’s very easy just to doubt myself when you encounter these problems many times in a row. And it can be very discouraging. It’s hard to get yourself out of that state. [2]*

Part of it is about personal responsibility, as a student does have agency in choosing whether or not to use online solutions. But when one choice overwhelms the status quo, it becomes a matter of choosing between using online solutions or getting left behind. When peer behavior and comparison is such a large driving factor, the ultimate effect of this culture is that it makes students question whether they belong in the course and in the major—whether they are capable of pursuing the discipline further. And these feelings can run deep:

> *I don’t know, I didn’t think I was dumb or anything, just…you know, I don’t know. It kind of sucks…Sometimes, you just don’t feel like that you should be there because you don’t know how to do something that everyone else knows. Just kind of sucks…Yeah. [7]*

This student had an extremely visceral reaction to this topic, and began crying after recounting what he had to overcome to continue in the math major.

As much as these kinds of conversations dominated interviews, I also want to acknowledge that two students believed this culture was beneficial to them. For one student, this culture empowered him to take control of his own learning, and allowed him to truly master proof-writing skills and the mathematical content. Online solutions were simply another resource to augment his learning and to be taken advantage of:

> *I don’t think it did affect my confidence to pursue math at all. In the end, I do think I really did learn a lot about proving just from this method. And I went into the final exam, where there were a lot of proof statements—and I think my professor was personally super good about her exams (she put harder questions on the homeworks and then the exam questions were more doable). Yeah, I feel like I learned enough throughout the year by doing this process, looking at people’s work, seeing how they did it, that I pretty much killed the exam. I was super happy. And I felt really good. I left the class like, I really truly feel like I understand all this material that I learned, and I felt like that was just super satisfying. So, I don’t think it affected it really. I think if anything, it made me feel more confident in my abilities. [4]*

The other student described himself as a “big person who just doesn’t let what other people are doing affect me very much,” and instead viewed the situation more as a challenge for him to tackle, finding others’ success to be motivating and something to aspire to [10]. The difference between these two students' positive responses and the negative ones before are stark, but I unfortunately do not have the data to draw any firm conclusions. I would hypothesize that individual factors likely contribute, as neither student was a first year at the time they took the class, which may have translated to a more mature and confident mindset.

A Culture of Collaboration

In addition to online solutions, students often turned to their peers when looking for a source of learning to supplement the inductive teaching method (this was part of approach 2 to homework). Either at TA office hours or within their own study groups, students would work collaboratively on the problem sets, bouncing ideas off of each other and discussing the merits of different approaches to the proofs. Working together with other students (and learning to embrace this collaborative approach) was a large component of most students' experience with the course, which I now identify as a thriving culture of collaboration:

> *Something that just came to mind was collaborating with others. I used to think that math was something that, for the most part, is relatively independent. You just solve a problem using what you know, and you're kind of just done with it. But especially with proof-writing in some more abstract and complex mathematical concepts, it's very helpful to have someone, if not a group, being with you and figuring out that situation together. I had a study buddy for the class, and it was—we did homework together, just thinking about how to approach problems was actually really fun because we could bounce ideas off of each other, and help each other in terms of how we define our proofs as well, now that I think about it, because people learn differently, and proofs are hard, so I think working with others on proofs in learning proofs is very helpful, especially if you don't know what you're doing and someone else does, someone else could structure that process for you. [2]*

Collaborative learning and inductive learning happen to be very complementary. While the behavior may have arisen to address shortcomings of the inductive teaching, a collaborative setting enables students to share observations and insights with each other, and discover concepts or rules in a more constructive, efficient manner. One student begins to hint at this:

> *But especially with proof-writing in some more abstract and complex mathematical concepts, it's very helpful to have someone, if not a group, being with you and figuring out that situation together. [2]*

Accordingly, and unlike expectations surrounding online resources, instructor expectations regarding collaboration were very clearly communicated to and understood by students. All of the course syllabi promoted collaboration, containing a statement to the effect of: "You are welcome and encouraged to collaborate with other students, however writing the final draft should be your own individual effort."[49] Students relayed as much, explaining that:

[49] Anibal Velozo, "Math 225 Spring Semester 2019, Section 01," (2019), https://yale.instructure.com/courses/45410.

> *There's a lot of collaboration, which I think the class does encourage between students, so I would work with a friend and we would try to come up with ideas. [10]*

One student, despite being aware that his instructor "[was] also very strongly in support of the collaborative process," expressed how he initially was unsure what collaboration actually entailed, because using other students' ideas could feel like crossing a line [9].

> *I remember that I'd asked a question about that at one point, because I thought to some extent, it felt kind of like cheating at times, because if I wasn't really able to do the problem, I could rely on the work of others to get me through it sometimes. And his response to that was that it's really a balance, ultimately, because usually there will be some of that, but then you also offset that by having your own ideas that you bring to the table. [9]*

This conversation shows that the instructor fully trusted the student to use his judgement and identify a balance that felt fair, but also that students could benefit from further clarification about what proper and improper collaboration each entails. Still, students were much more likely to be on the one same page as their instructor about collaboration than about internet sources. In addition, the explicit inclusion of "collaboration" on course syllabi shows how a culture of collaboration is promoted and in part constructed by the course itself.

Students found collaborative learning experiences to be overwhelmingly positive, in that they enhance not only their learning, but also their interest in the material and their confidence.

> *That [collaboration] ends up being really good for my confidence, in those group settings. Which is sort of counterintuitive to me in some ways, because again, I would expect that I would feel doing the problems on my own without any help, but practically, it's actually the other way around. Where being in a group setting working with TAs and peers is actually more fulfilling and boosting to my confidence, and working totally alone is can be pretty negative for my confidence in the end. [9]*

Multiple students expressed how the course was challenging and work-intensive, and working with peers on problem sets alleviated some of this burden:

> *It was hard and yeah, I mean I'm glad I took it. I think I learned a lot. I think it got better but it was also very hard and at times, maybe discouraging. But it was better to have peers near me and peer tutors who could help me, [3]*

In general, students "really enjoyed the collaborative learning experience," finding mental strength, motivation, and mathematical wisdom in their peers [10].

Becoming Proficient at Self-Learning

In the previous section *Teaching in the Classroom*, I discussed how students were often critical of their instructor's usage of class time because the pacing did not allow them to make the inductive observations that were crucial to their learning. They resorted to self-study to remedy this, "learning independently from the textbook," their notes, or others sources, to "figure out what was going on and [take] the time to understand proofs" [2]. Students who frequently took this route became proficient at it, developing a set of highly transferable skills:

> *I feel very comfortable in being able to look things up and then learn them by myself now, because that class kind of forced me to. I don't ever like searching up homework problems because I just don't feel comfortable doing that. But I know that I can search up proof techniques or I can search up similar problems to understand what techniques I'm supposed to use. [7]*

Importantly, in this quote the student expresses how he felt "forced" to learn how to write proofs on his own and become comfortable with self-learning, suggesting that he did not feel this element of self-study was part of the expectations for the course.

I am very interested in uncovering any discrepancy between students' learning and instructor expectations of their learning, and during interviews I asked each student: "Do you think you learned to write proofs in the same way you were expected to?" Unlike the previous two behaviors I discussed (usage of online resources and collaboration), learning is at the core of this course, and one would therefore expect instructors to clearly communicate the learning expectations of the course—what students should be learning, and how they should be learning it (and what self-study entails, if it should be part of their learning process). Unfortunately, in response to my question, most students were hesitant to make any claims about their instructor's expectations, because they were not certain what those expectations were. Said three different students: "I guess I don't know what she expected;" "I'm not entirely sure how they expected us to learn;" and "I don't know much about his expectations" [1, 2, 9]. The communication of learning expectations did not seem to be explicit, with only one course syllabus offering the advice of "take twice the amount of time of lectures to review what you learnt in class."[50]

Other students made inferences about their instructor's expectations. One believed their instructor expected students to learn how to write proofs during the Teaching Fellow's office hours, which for reference are not mandatory and occur in the evening separate from class:

> *I can only infer to the best of my ability [how they expected us to learn] given how they taught us. My guess is that they expected the TF session to help somewhat more, because they dedicated an entire session to teaching proofs, and she said if you want a head start you should go and check it out. But me, I don't think I personally got a lot out of—it was more like a show, or like something to just look at, instead of something that we had to learn and do ourselves the next day. [2]*

[50] You Qi, "Syllabus of Linear Algebra and Matrix Theory," (2016), https://yaleedu.sharepoint.com/sites/SyA/Syllabus%20Files/MATH22501(F16)_s28725.pdf.

Instead of being taught how to write proofs during the course lectures, this student was under the impression that his instructor had relegated the teaching of proof-writing to an external session held by a graduate student. Another student felt that his instructor wanted students to just “figure it out,” without offering more specific guidance:

> *I think this probably was his more expectation more or less, which was: you have a fundamental ability to reason and I’m giving you the conceptual knowledge to technically be able to figure it out with some insight. [5]*

In both instances, the learning expectations students believe their instructors have would be inappropriate for the teaching of a skill as foundational as writing proofs, suggesting again a disconnect between what instructors expect and what students perceive. One student also expressed there being a mismatch between her instructor’s expectations and reality, stating “I feel like she expected us to learn it faster than we did” [6]. This may have contributed to the pacing issues in the lectures.

Relatedly, some students felt that their instructors did not level students’ expectations for the course or for learning proof-writing appropriately. As one student puts it, “when you introduce a class as an introduction to proof-writing, you would think that there would be an introduction to writing proofs,” but by nature of the inductive teaching method this was not the case [10]. To another student, her instructor was not mindful that students could come from different backgrounds, which made the course feel less accessible in the beginning:

> *Just in the beginning, [the concepts] were all really abstract, and there was little recognition that it was abstract…and that it was new. And for some reason, it was taught like it wasn’t new. [1]*

The conversation has diverged slightly from self-learning to learning in general, but the two are closely intertwined in the context of instructor expectations: because self-driven learning was not always advertised as a critical part of learning proof-writing, students had to discover on their own just how necessary self-studying is, which can be burdensome and stressful. Students described the prospect and process of learning to write proofs in these courses as “daunting,” “intimidating,” and “more uncertain and less fun than it could have been” [10, 2, 5]. In most cases, it is not only the lack of clear expectations surrounding learning which contribute to student distress—if students believe their instructor expects them to become fully competent in writing proofs solely through lecture, while they must resort to copious self-study, then the culture of self-learning unfortunately translates to failure to meet expectations. Students are more empowered and have more agency when they possess "the belief that one is able to achieve the learning expectations of a course."[51]

Reflections on Inductive Learning

As the rest of this subsection has illustrated, students are keenly aware of their mental and emotional states in the context of a particular course. Using their stories, I identified trends in how the inductive teaching method affects student wellbeing and, more broadly, shapes course culture. To conclude this discussion, I now want to share students' *direct* reflections. Given their experiences and the cultures they participated in, what conclusions about inductive learning do they draw on their own?

Inductive learning is challenging. By nature, it places an enormous amount of responsibility on the student, and this burden can be exacerbated by insufficient guidance from the instructor. Not only must students ensure that they are discovering and inferring the *intended* concepts from their observations of examples, but they also can be tasked with sourcing examples if the exposure they gain from class is not enough. This can be frustrating, confusing, and unpleasant, as this student describes:

> *To that extent, I felt like there wasn't a lot of deductive ways to learn and inductive was the way to go, but I wasn't getting the inductive exposure from the class; I had to turn elsewhere, be it friends or the TF, or stuff floating on the web. And that made the process a little bit disorganized. So, I was doing inductive learning, which I think is probably necessary to some extent, but not in a way that was extremely well facilitated, retaining bits and pieces of insight for future problems. The final result of which was a less than very compelling 225 experience, where I felt like I had really internalized a lot of concepts and will be able to apply them later in my life to other classes. And just the general feeling that this sort of a class is more of a hassle than it's worth. Because you have to run around to collect all the answers. It's like Pokemon Go—you have to run around to collect answers and then assemble them in some cauldron. [5]*

At the same time, inductive learning is a valuable and integral part of the process when learning to write proofs. Especially for developing more creative skills, repeated and thoughtful exposure can be more helpful to building intuition than direct, deductive instruction. Students are aware of the progression from deductive to inductive, and see the necessity of each:

> *I maybe started learning deductively first, for a little bit, left it there, and what really helped was the inductive learning. [2]*

Learning inductively in itself is also a skill, and one that once developed allows students to exert greater control over their learning:

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[51] "Academic Integrity," UC Berkeley Center for Teaching & Learning, (2020), https://teaching.berkeley.edu/resources/design/academic-integrity.

> *I think being comfortable with how to learn inductively is very important, but it should be supplemented. [5]*

Especially when the quality of instruction is not consistent, capable inductive learners will know how to utilize the written resources at their disposal.

However, as important and necessary as inductive learning was for these students' when learning to write proofs, their experiences ultimately suggest to them that the inductive teaching method needs to be supplemented or altered. Whether this is by moving to a more balanced combination of deductive and inductive teaching of proof-writing,

> *[What I hoped, though, was to learn deductively first. Or at least, have a set of instructions to follow, before looking at examples and applying them. [2]]*

or by providing additional supports in the places where inductive learning occurs now, is a decision left to the instructor and the constraints imposed by time and departmental requirements on curriculum. Regardless, students do believe that it is possible to improve upon their experience, seeing as such models exist at other institutions:

> *I think it could have been better, but I don't know what other place I was supposed to go to learn these kinds of things. Because there's not a class here that teaches you proof writing, per se. Most the time, it's just kind of, you go and you just do it. I'm taking Discrete Math right now, and we've gone over induction, but at this point it's kind of too late [laughs]. So, that's kind of a place where you could learn proofs, but it's not really marketed that way. There's not really a place that's marketed that way. It was a little bit frustrating, especially because I have friends at other universities and they have classes just on proof writing, and we don't really have that. So, maybe it'd help to have some place where you could learn, at least, or know where to find the resources. [7]*

Recommendations to Instructors

In order to make the process of learning to write proofs more positive and tractable for students, instructors of introductory proof-based math courses should consider the following recommendations, which center around establishing clear course expectations and augmenting the inductive learning process:

1. Clearly communicate to students how they should expect to learn to write proofs. The curriculum for all introductory proof-based math courses at Yale is crafted around different content areas, and not different proof skills. Still, proof-writing is a skill that many students encounter for the first time in these courses. Instructors are responsible for making sure that these students are able to learn to write proofs within the confines of the course, and instructors must explicitly communicate their expectations so that students have a definitive answer to the question, "How are you expected to learn to write proofs in this course?"

So that students are aware of what exactly learning to write proofs entails, instructors should consider identifying the different proof components. Instructors must communicate the aspects of learning that will respectively be deductive or inductive, as well as both where the inductive exposure will come from and how much effort students should expect to expend studying material outside of class. Additionally, many instructors already emphasize that homework is important for learning. What proof-writing skills should students be learning from the homework? Should students be able to independently prove all of the homework problems, given the instruction they have received? If not, how should students approach homework problems that they are unable to solve?

Instructors must have answers to these questions, and deliberately communicate these answers to students early in the course, and perhaps written on the syllabus. All of this serves to increase students' abilities to evaluate whether they are achieving the learning expectations of the course, which empowers them to act in the best interest of their learning.

2. Establish a course policy regarding usage of online resources. Willfully ignoring the proliferation of online solutions usage is not an effective way of dealing with this culture. In fact, turning a blind eye or choosing not to explicitly address this behavior negatively impacts students' wellbeing and their ability to evaluate whether they are achieving the course's expectations. Because online solutions majorly bolster the inductive learning process, instructors should not bar their use. Instead, instructors must acknowledge that solutions to classic math problems are readily available on the internet, and state that they trust students to use these online resources to support their learning. Of course, students should remain committed to first trying to solve the proof on their own for a significant period of time. If a student still does not know how to begin a proof, it is okay to use an online solution to find a starting point. If a student is still stuck somewhere, it is okay to read an online solution, fully understand the point of difficulty, and reconstruct the proof in a thoughtful and effortful way. It is not okay to blindly copy a solution without understanding it.

Almost all students in an introductory proof-based math course are purposely electing to take an advanced class (over a non-proof-based one, for example), and as such are highly motivated to learn the material and progress to higher level coursework. Instructors must trust students to make choices that are best for their own learning, and communicate this to students.

If this recommendation is not immediately palatable, instructors should at least ensure that some policy regarding usage of online resources is explicitly communicated to students.

3. Acknowledge the struggle. Learning to write proofs can be very challenging, and instructors can foster a collaborative, nurturing environment by normalizing and acknowledging this fact. Students enter proof-based courses with a wide variety of backgrounds, and their preparation with proofs coming in often dictates their success early in the course. To struggle with the content or with the homework can be very discouraging for students, impacting their confidence. An instructor who reminds students that these feelings are common and valid, and that their struggle is a necessary part of the learning process, is one who makes mathematics more accessible.

4. Continue to promote collaboration among students. Collaboration complements inductive learning, and creates a positive, fun environment for learning mathematics.

5. Demonstrate and convey sociomathematical norms appropriate for the course. This was an ongoing point of difficulty for a few students. Especially in more foundational courses, conventions and properties of objects that can be taken for granted are frequently changing in the beginning, as more and more is learned. For instance, in a Linear Algebra course, once the curriculum moves from fields to vector spaces, the uniqueness of the additive identity is no longer something that students need to prove. It is not always clear to students the extent to which they must justify statements, and instructors can clarify sociomathematical norms by demonstrating them in class. One student who had learned to write proofs in PROMYS, a number theory summer camp, provides a detailed explanation of what this would look like:

> *PROMYS seemed to have a more streamlined approach to: What is a good proof? What tools do you have that you can use to prove things? Because I think we knew what we were allowed to use, whereas in 230, there was not as much understanding of: what assumptions can you make? What tools are you allowed to use? I think the instruction in PROMYS was definitely matched in the level of rigor that we needed. In lecture every day, we justified every single step that was made. [8]*

Instructors can also dedicate a period of class time to teaching towards the homework:

> *In Logic, we have a TA who at the end of every class does a quick, informal sort of section, but it's during class. He does a 25 minute "this is what I'm looking for in proofs." I think that's really helpful. [6]*

6. Ensure teaching assistants are on the same page regarding these expectations. Instructors must communicate their expectations for students to the tutors and teaching fellows, so that a consistent message can be delivered to students taking the course. This reduces confusion and ensures that students have similar expectations for their learning.

7. Provide students with additional examples of proofs tailored to the homework. Many students expressed that they did not receive sufficient exposure to the proof components needed on homework during the course lectures, and as such had a hard time extrapolating the different skills. Even if an instructor is very intentional about including good "examples" during lecture, the variety of proofs that can be covered during class are limited by time constraints. Instructors must juggle teaching new content, justifying theorems (sometimes involving more complex proofs than the ones on the homework), and proving statements similar to homework problems. To supplement the inductive learning of different proof skills, instructors can provide students with a set of 10-15 problems and their corresponding proofs. These proof sets would contain examples that the instructor regards as exemplary models of proof-writing, and ideally, they would be provided weekly and match in content to that week's problem set. Students then have the option of studying these curated examples over scouring the internet for additional proofs, and struggling students can use these as models for completing homework.

Consider the following example, which is copied directly from the first problem set of Math 225 (F19) and the corresponding provided solution:

> **Exercise.** Prove that the $\mathbf{0}$ in a vector space is unique.
>
> **Solution.** Let the vector space be V. Suppose there is $\mathbf{0}'$ such that for every element $x \in V$ we have
> $$x + \mathbf{0}' = x.$$
> In particular, we have
> $$\mathbf{0} = \mathbf{0} + \mathbf{0}' = \mathbf{0}'. \qquad \blacksquare$$

This problem requires an understanding of how to approach problems about "uniqueness," and a clever application of the zero axiom of vector spaces. The examples to share with students would exhibit these two key components, or use mathematical syntax and language that are applicable to this problem (and which may initially be difficult for students). Consider:

- Prove that the negative vector (additive inverse) of a given vector is unique.
- Prove that the zero vector multiplied by any scalar is the zero vector.
- Prove that the multiplicative identity in a field is unique.

The former two proofs both use the zero axiom, and the last one is essentially the same in structure as the desired proof. It also highlights the distinction between fields and vector spaces.

8. Create sets of hints for the homework problems. Students found the interactivity of teaching assistants' office hours to be very helpful to their learning (the second approach to homework). TAs can provide just the right amount of assistance to make a problem workable, while preserving the majority of the problem-solving for students to do. The hints that they provide are products of both their math expertise and their understanding of the needs of the individual student, but instructors can emulate this process to an extent by creating similar hints for students to follow. Especially because not all students can attend TA office hours, providing such a resource empowers students to derive more of the proof on their own.

The majority of proofs in an introductory proof-based math course are not too involved, and hence usually require only a few key insights—using a particular definition or theorem, cleverly applying some property, identifying a relation between two things, etc. To write a proof, one must have arrived at these insights, and there is only a small jump needed to turn these insights into prodding questions or a suggestion. For the previous problem, this would be:

1. What does it mean for something to be unique?
    a. It means there can't be two truly different elements which satisfy the properties of that element.
    b. Try assuming there is another one.
2. Suppose there is another element, $\mathbf{0}'$, which follows the definition of the zero vector. What does it mean for $\mathbf{0}$ to be unique?
    a. It means that $\mathbf{0}$ and $\mathbf{0}'$ are the same, that they're equal.
    b. Can you use the definitions to arrive at this equality?
3. What are the definitions of $\mathbf{0}$ and $\mathbf{0}'$?
    a. The definitions are, for any $v \in V$, $\mathbf{0} + v = v$, and $\mathbf{0}' + v = v$.
    b. What is $\mathbf{0} + \mathbf{0}'$?

This kind of guided walkthrough helps motivate the problem-solving, and allows students to gradually use more assistance as they determine is necessary. Even though the creation of such hints can take time, which is an added responsibility for the instructor or TA, they give students independence and control—crucial in developing mathematical confidence.

9. Provide solutions to all problem sets and exams. Inductive learning is an effortful process. To extract any useful insight, students must carefully observe and thoughtfully process the “example” proofs that are provided to them. This includes everything from the proof’s structure to the syntax and formalism, and beyond comprehending the proof’s logic, students must also figure out why that particular path of logic was chosen. Students are able to perform this kind of analysis—and subsequently this kind of inductive learning—much more efficiently and effectively for proofs that they have already attempted to construct, on the homework or on an exam. Perhaps there was a certain part of the proof they struggled with, or perhaps they were not certain their technique was the best one to implement. Reading instructor-certified solutions can help students verify or adapt their approach to different types of proofs.

Instructors should post solutions which strive to articulate as much of the author’s thinking as possible. Unfortunately, within the field it is common to equate mathematical elegance to brevity, or to idealize proofs that are almost magically inventive, which use ideas that seem come out of nowhere. The proof example I provided, from the solutions document to the first problem set of Math 225 (F19), ascribes to these ideals—it is extremely brief, and opts to use phrases like “we have” instead of articulating the motivation needed to arrive at such a conclusion. Proofs like these may reflect the sociomathematical norms followed by most professional mathematicians, but they are not as accessible and useful as they could be to students beginning to learn proofs. In the case that the instructor does not have the time or resources to write solutions, they may consider compiling solutions from the work that students have submitted.

Limitations

The data sample I drew from was the primary limiting factor in this study. As I mentioned in the Methodology, 12 student interviews cannot capture the breadth and diversity of all student experiences in introductory proof-based math courses. The missing voices I discuss here represent interesting future directions to expand this study.

Due to the snowball method I used to recruit interviewees, the students I spoke with all happened to be part of a subpopulation that (1) frequents teaching assistants' office hours and (2) is or was fairly successful in Math 225 or Math 230. I tried to select students who were diverse along criteria like class year and major, but because the snowballing started with me (a peer tutor for Math 225 for three semesters), the people I was able to reach tended to belong to a network of students who take full advantage of their peer tutors, ULAs, and TFs. In reality, a significant number of students in Math 225 and Math 230 may never attend such office hours. The interactivity and guidance available at such sessions (which I claimed were central to the inductive learning process) then do not contribute to how these students learn proofs. This means the responses I received are biased towards this kind of "interactive guidance," and there is perhaps not universally as much value to this resource as I deduced from my interviews.

The students willing to participate in interviews are also ones that were fairly successful in the course. This is not to say that they did not find the experience difficult or that they did not struggle, but that they endured despite the challenge—making it through the introductory class with enough of a foundation and inclination towards math to pursue more advanced coursework. My capstone illustrates the experiences of highly-motivated and high-achieving students, who are interested in speaking about and reflecting on their learning process. My capstone does not capture the perspectives of students who felt so discouraged or put off that they chose to leave math behind entirely or simply dropped the course. Furthermore, the thoughtfulness and intentionality behind the approach that my interviewees took to learning is likely not the norm.

Because I only interviewed students at Yale College, the nature of learning and the behaviors I discuss in this capstone are in some ways predicated on the culture and teaching norms of the Yale Math Department. Students who attend institutions with bigger math programs, with more introductory courses taught by tenure track faculty, etc., may have radically different experiences.

The biggest piece from the narrative of this capstone are the voices of the instructors who design and teach these introductory proof-based math courses. Over informal conversations with Yale's math faculty this past semester, I have learned that many decisions about these courses—what is taught, how they are taught, whether they are taught—are governed by history and precedent. Questions which are worthwhile to answer: What is the history of Math 230 at Yale? What are the dynamics of the opposing sides (for a proofs vs. proof-based course) inside the Yale Math Department? What role do non-tenure track faculty, who teach the majority of undergraduate Yale math courses, have in making decisions about curriculum and teaching?

And beyond shedding light on the institutional forces that shape student experiences, instructors' voices provide a window into the teaching side of writing proofs. To make sense of their reasoning for teaching the way that they do, to understand the changes they would be receptive to—this is the way to progress.

Conclusion

This capstone finds that learning to write proofs is a primarily inductive process. While more straightforward proof components can be taught effectively using deductive methods, the aspects of proofs which make them challenging—the approach, the structure, and the intuition—are best learned inductively. Students obtain the exposure and examples necessary for effective inductive learning largely through the course homework. In particular, it is the homework problems that students are *unable* to prove that facilitate learning the most. By working off of hints from teaching assistants and peers or by reconstructing a solution found on the internet, students develop new insights. Rather than viewing this as copying or cheating, I found that these methods help students gain exposure to the approaches and reasoning of more experienced mathematicians, especially at the critical points where they may be struggling, all of which is very conducive to inductive learning.

An inductive approach to teaching has certain consequences on student behavior and course culture, particularly when it is not done well. Because the inductive method places greater responsibility on the student, many become proficient at self-learning and self-study, and others make a habit of working collaboratively with their peers. These are positive and transferable skills. On the other hand, when the inductive teaching is not sufficiently supplemented with examples of proofs for learning to take place, students turn to the internet. This creates a culture of overdependence on online solutions, which can be discouraging and harm student confidence.

These effects can be ameliorated in part with more guidance and clearer expectations from the instructor. I recommended that instructors lay out clear expectations regarding learning and online resources, and also provide students with supplemental resources for completing problem sets, which is where inductive learning is particularly effective.

It is my hope that this perspective—one which bridges mathematical proofs and inductive learning—can help instructors of introductory proof-based courses empower their students and make advanced mathematics more accessible. But ultimately, even if this is not the case, even if the recommendations that I proposed are not palatable to instructors or are otherwise impossible to achieve, this paper shares the stories of thirteen students—the twelve I interviewed, and my own. Studying math has been one of the most challenging and rewarding experiences I've had at Yale, and I hope that more than anything else, the experiences and stories in this paper remind students that math can be hard; that it is normal to struggle; and that in the end, it's worth it.

Acknowledgements

Thank you to:

- My friends—I have been so confused, and frustrated, and exhausted throughout this process; thank you for being my sounding boards and my anchors, for work zooming with me and for listening to me complain. Jason Chen and Alex Wang, thank you for being my reason for continuing with math these past four years.
- My family—Everything I care about, I care about because of you. Thank you.
- Otter.ai—I don't think I could have transcribed ten hours of audio without you.
- My Being Human in STEM cohort—My capstone started with our project.
- Professor Asher Auel—I wouldn't be a math major today if I hadn't been so lucky to stumble into your 225 course my first-year spring. When I was most ready to give up on math, you convinced me to stay.
- Professor Talya Zemach-Bersin and Anne Mishkind, Professor Mira Debs—Thank you for giving me the chance to write and do something I never expected I could.
- Professor Patrick Devlin—If all instructors were like you, this kind of capstone wouldn't be necessary. Thank you for being so receptive and generous with your time. Your guidance kept me grounded; our conversations kept me focused on the things that matter.
- Professor Richard Hersh—This capstone project would not exist without you, and I am so grateful to have had your mentorship. Thank you for your feedback and your patience, for motivating me to write better, for challenging me to think harder. I owe this all to you.

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Appendices

Appendix A: Semi-Structured Interview Script

Introduction: background and context

- Did you know how to write proofs before Math 225/230?
- What was your preparation with math coming into college from high school?
- I'm interested mainly in hearing about two things today: first, how you believe you learned to write proofs, specifically how you used inductive learning (which we'll define later) in that process; and second, the discrepancies between how you were taught and how you learned, and how these affected your experience. I'm interested in your story and your narrative. One thing I want to point out is that my project is focused on proof-writing as a skill, and not necessarily the mathematical content itself.
- According to whatever this means to you, do you currently know how to write a proof? [*expect answer of yes*] And prior to your first experience with proof-based math, your answer was no. During this interview, I will be asking you to reflect on the process and actions that allow you to now give a different answer to the question, "do you know how to write a proof?" I have my own answer to this question, and I hope we can be as candid with each other as possible. At the end of this interview, we will revisit this question, but in a few sentences, can you tell me: how do you think you learned how to write a proof?

Challenges of learning proofs

- What are the different components of writing proofs that you needed to learn?
- *Provide [Prompts for identifying proof-writing components]*. Feel free to reference the proofs provided; these came from the first 3 problem sets of Math 225.
- Of the components we identified, which ones did you struggle with or were challenging for you to learn? Why? What made them challenging?

Expectations and inductive learning

- For the components we identified, how did you learn those proof-writing skills?
  - And how were you taught these skills? How were you expected to learn?
  - Do you think you learned in the same way you were expected to learn?
- My project tries to view learning through the lens of inductive and deductive learning. Inductive learning is a process where the student discovers rules by observing examples. In contrast, in deductive learning, students are given rules that they then need to apply.
- For these components, do you think you learned deductively? Inductively? How so?

The role of homework

- When doing homework, you have access to a variety of resources. Can you name resources offered by the class, and those external to the class, which you used?
- For proofs on the homework, do you feel you *derive* all the solutions on your own?
  - [*continue this line of questioning if the answer is no*]
  - And what is it like for other problems? Independently, with some small hints, working with other people. Distinction between proofs where you derive, for the most part, the solution on your own, and those where substantial parts of the problem are revealed to you (by a friend, peer tutor, internet, etc.).
- Do you think there is one approach that is "better" to take than the other? Why?
- You mentioned you did not always take the approach which you believe is better. What were some reasons why you did that?

- How do you think taking that approach impacted your learning, specifically in regards to learning to write proofs?
- How do you think taking that approach impacted you emotionally? Did it impact your confidence or interest in pursuing mathematics further?

Summary

- Let's revisit the question from the beginning. You learned how to write proofs in Math 225/230. What changed over the course of the semester, and what was your learning like, intellectually and emotionally?

Sample problems:

1.5. Prove that the $\mathbf{0}$ in a vector space is unique.

Solution. Let the vector space be V. Suppose there is $\mathbf{0}'$ such that every element $x \in V$ we have

$$\mathbf{x} + \mathbf{0}' = \mathbf{x}.$$

In particular, we have

$$\mathbf{0} = \mathbf{0} + \mathbf{0}' = \mathbf{0}'.$$

4. Let W_1 and W_2 be two non-empty subsets of a vector space V. Then we define the sum of W_1 and W_2, denoted $W_1 + W_2$, to be the set $\{x + y : x \in W_1, y \in W_2\}$. Now let W_1 and W_2 be subspaces of V.

(a) Prove that $W_1 + W_2$ is a subspace of V that contains both W_1 and W_2.

(b) Prove that if W is a subspace of V that contains W_1 and W_2, then it also contains $W_1 + W_2$.

Solution. (a) To show that $W_1 + W_2$ contains W_1 and W_2, we show that for all $v \in W_1$ and $u \in W_2$, $v \in W_1 + W_2$ and $u \in W_1 + W_2$.

Let $v \in W_1$ and $u \in W_2$. Since W_1 and W_2 are subspaces of V, $0 \in W_1$ and $0 \in W_2$. Then we can rewrite v as an element in $W_1 + W_2$:

$$v = v + 0 \in W_1 + W_2$$

since $v \in W_1$ and $0 \in W_2$. Similarly,

$$u = 0 + u \in W_1 + W_2$$

since $0 \in W_1$ and $u \in W_1$. Since v, u are arbitrary elements in W_1 and W_2 respectively, we have shown that any arbitrary element in W_1 or W_2 is contained in $W_1 + W_2$. Thus $W_1, W_2 \subseteq W_1 + W_2$.

(b) Suppose W is a subspace of V that contains W_1 and W_2. We want to show that every vector x in V that can be written as $u + v$, where $u \in W_1$ and $v \in W_2$, also belongs to W. Let x be an element in V such that it can be written as $u + v$, where $u \in W_1$ and $v \in W_2$. Since $u \in W_1$ and W_1 is a subspace of W, $u \in W$. Similarly, since $v \in W_2$ and W_2 is a subspace of W, $v \in W$. Since $u, v \in W$ and W is a subspace of V, addition must be closed in W; that is, $x = u + v \in W$, as desired.

Appendix B: Email Request for Interview

Email address: joy.qiu@yale.edu
Email subject: Interview Request for Education Studies Project About Writing Math Proofs

Dear [interview subject]:

My name is Joy Qiu, and I am a senior in the Education Studies program. For my senior capstone project, I am conducting a study on how students in advanced university mathematics learn how to write proofs, with a focus on centering the student experience and examining the role of deductive learning in this process. I'm hoping that reflections from students like yourself may be used to inform the development of problem sets and other pedagogical practices within introductory proof-based math classes.

I was given your contact information by [reference] because you have experience with an introductory proof-based math class at Yale (either Math 230 or Math 225), and am reaching out to see if you may be interested in participating in a 30-minute interview.

The audio from the interview will be recorded, and the only identifying information I will be collecting will be (1) whether you are majoring in mathematics, (2) whether you are a junior or senior, and (3) whether you are or were enrolled in Math 230 or Math 225 in the last 6 months.

Would you be interested in participating in this interview?

Thank you in advance and I look forward to hearing from you!
Joy Qiu

Appendix C: Verbal Consent Form

Verbal Informed Consent Script for Participation in a Research Study
HSC #2000026874

Hi, my name is Joy Qiu, and I am an undergraduate from Yale. I am conducting a research study to examine how students learn proofs in advanced university mathematics, focusing on the student experience and the role of deductive learning. Participation in this study will involve an approximately 30-minute interview.

As part of the interview, you will be asked to describe how you would solve a given set of math problems, and may experience frustration or discomfort during this process. There are no other known or anticipated risks to you for participating other than a possible loss of confidentiality. Although this study will not benefit you personally, we hope that our results will add to knowledge about best pedagogical practices in teaching students how to write proofs.

All of your responses will be held in confidence. Only the researchers involved in this study and those responsible for research oversight will have access to the information you provide. With your permission, your responses will be audiotaped and then transcribed, and stored electronically on a password-protected device. Your responses will be linked only to the following identifying information: whether or not you are a student majoring in math, whether or not you are a junior or senior, and whether or not you were enrolled in Math 225 or Math 230 in the last 6 months.

Participation in this study is completely voluntary. You are free to decline to participate, to end participation at any time for any reason, or to refuse to answer any individual question without penalty. Your decision whether or not to participate in this study will not affect your relationship with your Math 225 or Math 230 professor, your grades or your academic standing at Yale University.

If you have any questions about this study, you may contact the investigator, Joy Qiu, at joy.qiu@yale.edu or 630-788-6748. If you would like to talk with someone other than the researchers to discuss problems or concerns, to discuss situations in the event that a member of the research team is not available, or to discuss your rights as a research participant, you may contact the Yale University Human Subjects Committee, 203-785-4688, human.subjects@yale.edu. Additional information is available at https://your.yale.edu/research-support/human-research/research-participants/rights-research-participant.

Do you have any questions at this time? Would you like to participate in the study? Are you 18 years of age or older? Do I have your permission to record our interview?

Appendix D: Moore's Model

Model of major sources of students' difficulties in doing proofs (1994):